Supplementary Appendix to "The Bubble Game: An Experimental Study of Speculation"

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Abstract

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1 A theory of rational bubbles

The objective of this section is to show that bubbles can emerge in a financial market with perfectly rational traders and finite trading opportunities, without asymmetric information on asset payoffs. This theory is at the basis of the experimental design that we use in the main paper in which bubbles can be rational or not depending on whether or not there is a cap on prices. In this Supplementary Appendix, the theoretical analyses, presented in this Section as well as the next two ones, focus on the case of self-financed traders because their individual rationality constraints are very close to financiers' ones in the trader/financier game used in the experiment presented in the main paper.

Consider a financial market in which trading proceeds sequentially. There are T agents, referred to as traders. Traders' position in the market sequence is random with each potential ordering being equally likely. Traders can trade an asset that generates no cash flow and this is common knowledge.[1] This enables us to unambiguously define the fundamental value of the asset: it is zero in our case because, if the asset cannot be resold, an agent would not pay more than zero to buy it.

The asset is issued by agent 0, referred to as the issuer.[2] The first trader in the sequence is offered to buy the asset at a price P_1. If he does so, he proposes to resell at price P_2 to the second trader. More generally, the t-th trader in the sequence, $t \in \{1, ..., T-1\}$, is offered to buy the asset at price P_t and resell at price P_{t+1} to the $t+1$-th trader. Traders take the price path as given, with $P_t > 0$ for $t \in \{1, ..., T\}$. Finally, the last trader in the sequence is offered to buy the asset at price P_T but cannot resell it. If the t-th trader buys the asset and is able to resell it, his payoff is $P_{t+1} - P_t$. If he is unable to resell the asset, his payoff is $-P_t$. For simplicity, we consider that if a trader refuses to buy the asset, the market process stops.

We consider that traders are risk neutral. We show in the next section that our results hold with risk averse traders. Individual i has an initial wealth denoted by W_i, $i \in \{1, ..., T\}$.[3] As a benchmark, consider the case in which traders have perfect information, that is, each trader i knows that his position in the sequence is t and this is common knowledge. In this

[1]The asset cash flow could be positive and risky without changing our results.

[2]The potential bubbles that may arise in our environment can be interpreted as Ponzi schemes, and the issuer of the asset as the scheme organizer.

[3]In our model, traders might end up with negative wealth.

perfect information benchmark, it is straightforward to show that no trader will accept to buy the asset except at a price of 0 which corresponds to the fundamental value of the asset. Indeed, the last trader in the queue, if he buys, ends up with $W_T - P_T$ which is lower than W_T. Since he knows that he is the last trader in the queue, he prefers not to trade. By backward induction, this translates into a no-bubble equilibrium. This result is summarized in the next proposition.

Proposition 1 *When traders know their position in the market sequence, the unique perfect Nash equilibrium involves no trade.*

Let's now consider what happens when traders do not initially know their position in the market sequence, and this is common knowledge. We model this situation as a Bayesian game. The set of players is $\{1, ..., T\}$. The set of states of the world is Ω which includes the $T!$ potential orderings. ω refers to a particular ordering. The set of actions is identical for each player i and each position t and is denoted by $A = \{B, \emptyset\}$ in which B stands for buy and $\emptyset$ for refusal to buy. Denote by $\omega_t^i \subset \Omega$ the set of orderings in which trader i's position in the market sequence is t. The set of signals that may be observed by player i is the set of potential prices denoted by P. The signal function of player i is $\tau(i) : \omega_t^i \rightarrow P_t$, in which P_t refers to the price that is proposed to the t-th trader in the market sequence. The price path P_t is defined as follows. The price P_1 proposed to the first trader in the sequence is random and is distributed according to the probability distribution $g(.)$ on P.[4] Other prices are determined as $P_{t+1} = f(P_t)$, with $f(.) : P \rightarrow P$ being a strictly increasing function that controls for the explosiveness of the price path. A strategy for player i is a mapping $S_i : P \rightarrow A$ in which $S_i(p)$ indicates what action is chosen by player i after observing a price p. Conditional on observing $p = P_t$, player i understands that the next player j in the market sequence observes $f(P_t)$, and that he chooses $S_j(f(P_t))$. Using the signal function, players may learn about their position in the market sequence. A strategy profile $\{S_1^*, ..., S_T^*\}$ is a Bayesian Nash equilibrium if the following individual rationality (IR) conditions are satisfied:

$$\mathbb{E}\left[\pi\left[S_i^*(P_t), S_j^*(f(P_t))\right]|P_t\right] \geq \mathbb{E}\left[\pi\left[S_i(P_t), S_j^*(f(P_t))\right]|P_t\right],$$

[4]One can consider that this first price P_1 is chosen by Nature or by the issuer according to a mixed strategy characterized by $g(.)$.

$$\forall (i,j) \in \{1,...,T\} \times \{1,...,T\} \text{ with } j \neq i, \text{ and } \forall P_t \in P.$$

$\pi \left[S_i \left(P_t \right), S_j^* \left(f \left(P_t \right) \right) \right]$ represents the payoff received by the risk-neutral player i given that he chooses action $S_i \left(P_t \right)$ and that other players choose actions $S_j^* \left(f \left(P_t \right) \right)$. Remark that agents' payoff not only depend on others' actions but also on the state of nature because it is possible that they are last in the market sequence.

We now study under what conditions there exists a bubble equilibrium $\{S_1^* = B, ..., S_T^* = B\}$. The crucial parameter a player i has to worry about in order to decide whether to enter a bubble is the conditional probability to be last in the market sequence, $\mathbb{P} \left(\omega \in \omega_T^i | P_t \right)$. The IR condition can be rewritten as:

$$\left(1 - \mathbb{P} \left[\omega \in \omega_T^i | P_t \right] \right) \times \left(W_i + f \left(P_t \right) - P_t \right) + \mathbb{P} \left[\omega \in \omega_T^i | P_t \right] \times \left(W_i - P_t \right) \geq W_i,$$
$$\forall i \in \{1,...,T\} \text{ and } \forall P_t \in P.$$
$$\Leftrightarrow \left(1 - \mathbb{P} \left[\omega \in \omega_T^i | P_t \right] \right) \times f \left(P_t \right) \geq P_t,$$
$$\forall i \in \{1,...,T\}, \text{ and } \forall P_t \in P.$$

If $\mathbb{P} \left[\omega \in \omega_T^i | P_t \right] = 1$ for some i and some P_t, the IR condition is not satisfied and the bubble equilibrium does not exist. This is for example the case when the support of the distribution $g \left(. \right)$ is bounded above by a threshold K. Indeed, a trader upon observing $P_t = f^{T-1} \left(K \right)$ knows that he is last and refuses to trade. Backward induction then prevents the existence of the bubble equilibrium. The IR function is also not satisfied if the signal function $\tau \left(i \right)$ is injective. Indeed, by inverting the signal function, players, including the one who is last in the sequence, learn what their position is. These results are summarized in the following proposition.

Proposition 2 *The no-bubble equilibrium is the unique Bayesian Nash equilibrium if i) the signal function is injective, ii) the first price is randomly distributed on a support that is bounded above, iii) the price path is not explosive enough, or iv) the probability to be last in the market sequence is too high.*

We now propose an environment where the IR condition derived above is satisfied. Consider that the set of potential prices is defined as $P = \{m^n \text{ for } m > 1 \text{ and } n \in \mathbb{N}\}$, that is, prices are positive powers of a constant $m > 1$. Also, assume that $g \left(P_1 = m^n \right) = \left(1 - q \right) q^n$, that is, the power n follows a geometric distribution of parameter $q \in (0, 1)$. Finally, we set $f \left(P_t \right) = m \times P_t$. If there are T players on the market, the probability that a player i is last in the sequence, conditional on the price P_t that he is proposed, is computed by Bayes' rule:

$$\mathbb{P}\left[\omega \in \omega_T^i | P_t = m^n\right] = \frac{\mathbb{P}\left[P_t = m^n | \omega \in \omega_T^i\right] \times \mathbb{P}\left[\omega \in \omega_T^i\right]}{\mathbb{P}\left[P_t = m^n\right]}$$
$$= \frac{(1-q)\, q^{n-(T-1)} \times \frac{1}{T}}{\sum_{j=n-(T-1)}^{j=n} (1-q)\, q^j \times \frac{1}{T}} = \frac{1-q}{1-q^T} \text{ if } n \geq T-1,$$

and

$$\mathbb{P}\left[\omega \in \omega_T^i | P_t = m^n\right] = 0 \text{ if } n < T-1.$$

Under our assumptions, Bayes' rule implies that the conditional probability to be last in the market sequence is 0 if the proposed price is strictly smaller than m^{T-1}, and $\frac{1-q}{1-q^T}$ if the proposed price is equal to or higher than m^{T-1}. This conditional probability thus does not depend on the level of the price that is proposed to the players.[5] The IR condition can be rewritten:

$$\left(\frac{q-q^T}{1-q^T}\right) \times m \geq 1.$$

This condition is less restrictive when there are more traders present on the market.

There thus exists an infinity of price paths characterized by $m \geq \frac{1-q^T}{q-q^T}$ that sustain the existence of a bubble equilibrium. Obviously, there always exists a no-bubble equilibrium.[6] Indeed, if players anticipate that other players do not enter the bubble, then they are better off refusing to trade. These results are summarized in the next proposition.

Proposition 3 *If i) the T traders are equally likely to be last in the market sequence, ii) the price P_1 proposed to the first trader in the sequence is randomly chosen in powers of m according to a geometric distribution with parameter q, and iii) $P_t = m \times P_{t-1}$, $\forall t \in \{2, ..., T\}$, there exists a bubble Bayesian Nash equilibrium if and only if $m \geq \frac{1-q^T}{q-q^T}$. There always exists a no-bubble equilibrium.*

Our results hold even if one introduces randomness in the underlying asset payoff, and (potentially random) payments at interim dates. In the

[5] We implicitly assume here that players cannot observe if transactions occured before they trade. However, we do not need such a strong assumption. For example, if each transaction was publicly announced with a probability strictly smaller than one, our results would still hold. This probability should be small enough so that the likelihood of being last in the sequence is not too high.

[6] When there exists a bubble-equilibrium in pure strategies, there can also exist mixed-strategies equilibria in which traders enter the bubble with a positive probability that is lower than 1. We have characterized these equilibria for the two-player case. They involve peculiar evolutions of the probability to enter the bubble depending on the price level that is observed. We thus do not use these mixed-strategy equilibria in our analysis.

next section, we show that our results hold if traders are risk averse. One could be tempted to interpret our results as an inverse-Hirshleifer effect: going from perfect to imperfect information seems to imply a creation of gains from trade in our setting even with risk-neutral agents. However, note that it is not possible to compute the ex-ante welfare created by the game of imperfect information. Indeed, the expected payoffs of the players are infinite. To see this, remark that these expected payoffs are equal to:

$$\lim_{x \to +\infty} \left[\frac{m-1}{2} + \frac{m\,(m-1)}{4} + \left(\frac{q-q^T}{1-q^T}\,(m-1) - \frac{1-q}{1-q^T} \right) \sum_{n=2}^{n=x} q^{n+1} m^n \right].$$

This limit converges if and only if $qm < 1$ (see Section 3). This inequality is in conflict with the IR condition according to which $m \geq \frac{1-q^T}{q-q^T}$. This implies that the only games in which the ex-ante welfare is well-defined are the games where only the no-bubble equilibrium exists. This makes it hard to conclude that the imperfect information game is actually creating welfare even if interim (that is, knowing the proposed price), all traders are strictly better off entering the bubble if they anticipate that other traders are also going to do so. A more extensive analysis of welfare in the bubble game is offered in Section 3 of this Supplementary Appendix.

2 Bubble equilibrium with risk aversion

Consider the environment in which a bubble-equilibrium exists when players are risk-neutral. We now show that a bubble equilibrium can still exist if players are risk averse. The environment is as follows. There are T players. The set of potential prices is defined as $P = \{m^n \text{ for } m > 1 \text{ and } n \in \mathbb{N}\}$. The price that is proposed to the first trader P_1 is randomly determined following a geometric distribution: $g(P_1 = m^n) = (1-q)\, q^n$ with $q \in (0,1)$. Finally, the price path is defined as $P_{t+1} = m \times P_t$ for $t \in [1, ..., T-1]$.

2.1 Piecewise linear utility function

For simplicity, we assume that utility functions are piecewise linear with a kink at agents' initial wealth, that is player i's utility function is: $U_i(x) = x\mathbb{1}_{x\leq W_i} + [W_i + (1-\gamma_i)(x - W_i)]\,\mathbb{1}_{x>W_i}$, where $\gamma_i \in\ ]0{,}1]$ is a measure of player i's risk aversion. Conditional on trader i expecting other traders to buy, the IR condition is now written as:

$$\begin{array}{l}\left(\begin{array}{c}(1-\mathbb{P}[\omega=\omega_T^t|P_t])\times U_i(W_i+f(P_t)-P_t)\\ +\mathbb{P}[\omega=\omega_T^t|P_t]\times U_i(W_i-P_t),\end{array}\right)\geq U_i(W_i)\\ \forall i\in\{1,...,T\}, \text{ and } \forall P_t\in P\end{array}$$

$$\begin{array}{l}\Leftrightarrow\left(\begin{array}{c}(1-\mathbb{P}[\omega=\omega_T^t|P_t])\times[W_i+(1-\gamma_i)(f(P_t)-P_t)]\\ +\mathbb{P}[\omega=\omega_T^t|P_t]\times(W_i-P_t)\end{array}\right)\geq W_i,\\ \forall i\in\{1,...,T\}, \text{ and } \forall P_t\in P\end{array}$$

$$\begin{aligned}&\Leftrightarrow\gamma_i\leq 1-\frac{\mathbb{P}[\omega=\omega_T^t|P_t]\times P_t}{(1-\mathbb{P}[\omega=\omega_T^t|P_t])(f_t(P_t)-P_t)}, \forall i\in\{1,...,T\}, \text{ and } \forall P_t\in P\\ &\Leftrightarrow\gamma_i\leq 1-\frac{(1-q)}{(q-q^T)(m-1)}, \forall i\in\{1,...,T\}.\end{aligned}$$

This inequality indicates that, if players are not too risk averse, there exists a bubble equilibrium. Furthermore, the IR condition must hold for all traders, that is, all trader must be not too risk averse. Consequently, uncertainty about other traders' risk aversion may reduce the incentives to enter into bubbles, as trader i may expect the IR condition of following trader not to be satisfied because they would be too risk averse. Finally, when m gets larger, the range of risk aversion for which a bubble equilibrium exists is larger.

2.2 CRRA utility function

We now check that this results holds if utility functions are CRRA, that is player i's utility function is: $U_i(x) = \frac{1}{1-\theta_i} x^{1-\theta_i}$ if $\theta_i > 0$ and $U_i(x) = ln(x)$ if $\theta_i = 1$, where θ_i is a measure of player i's relative risk aversion. Let us assume that the trader's initial wealth W_i is larger than the price at which he is offered to buy. For simplicity, we assume that $W_i = P_t$.[7] For $\theta_i \neq 1$, the IR condition is now written as:

$$\begin{pmatrix} (1 - \mathbb{P}[\omega = \omega_T^t | P_t]) \times U_i (W_i + f(P_t) - P_t) \\ +\mathbb{P}[\omega = \omega_T^t | P_t] \times U_i (W_i - P_t) \end{pmatrix} \geq U_i (W_i), \\ \forall i \in \{1, ..., T\}, \text{ and } \forall P_t \in P$$

$$\Leftrightarrow \begin{pmatrix} (1 - \mathbb{P}[\omega = \omega_T^t | P_t]) \times \frac{1}{1-\theta_i} (W_i + f(P_t) - P_t)^{1-\theta_i} \\ +\mathbb{P}[\omega = \omega_T^t | P_t] \times \frac{1}{1-\theta_i} (W_i - P_t)^{1-\theta_i} \end{pmatrix} \geq \frac{1}{1-\theta_i} (W_i)^{1-\theta_i}, \\ \forall i \in \{1, ..., T\}, \text{ and } \forall P_t \in P.$$

$$\Leftrightarrow \theta_i \leq 1 - \frac{\ln(\frac{1-q^T}{q-q^T})}{\ln(m)}, \forall i \in \{1, ..., T\}.$$

The nature of this inequality is similar to the piecewise linear case.

[7]To see that this assumption does not change the probabilistic set up of the game, consider that the organizer of the bubble game first draws the trading prices and then picks the players appropriately according to their level of wealth. This requires that potential wealth be infinite and that there is at least one agent for each level of wealth.

3 Welfare analysis

This section shows that, when there is no price cap, the ex-ante expected welfare (before observing the offered price) is not defined in the bubble game at the bubble equilibrium. In this case, an agent with a non-bounded utility function could not decide, ex-ante (that is, before being proposed a price), whether playing this game is desirable or not. However, it is important to remark that, interim (that is, as soon as agents are being proposed a price), it is perfectly possible to compute the expected welfare (which is now finite and positive). This implies that, if one were to create a Ponzi scheme that follows our bubble game spirit, it would be optimal for this person to propose agents to play the game after describing the rules of the game and proposing a price at which they can buy. This is exactly what we do in the experiment. As a result, consistently with the treatment of Bayesian Games offered by Osborne and Rubinstein (1994, page 26), in any given play of the game, each player knows his type (that is, the price he is offered) and does not need to compute his ex-ante welfare. Consequently, the bubble game is a well-defined Bayesian game. When there is a price cap, expected welfare is well-defined both ex-ante and interim, and is negative.

Our game is related to the Super-Petersburg paradox of Menger (1934) as discussed, for example by Samuelson (1977), and to the two-envelope problem when the expected dollar amount is infinite as discussed by Geanakoplos (1992). In these two games, if participation is subject to a finite charge, expected welfare is infinitely positive. Players would thus agree to play these games. In the bubble game, the situation is a little different. Before being proposed a price, players cannot determine whether the game is worth playing because it involves comparing infinitely positive and negative payoffs. However, after being proposed a price, expected utility can be computed and might be positive, leading players to be willing to participate. Another difference between the bubble game and the Super-Petersburg game is the coordination of beliefs among players that must be achieved to reach the equilibrium. This is similar to the two-envelope game in which it might be profitable to switch an envelope only if the other player also switches.

3.1 Expected gains

We first show that without risk aversion, the ex-ante expected gains (before observing the offered price) can be positive or negative depending on how one computes conditional expectations.

We denote by $\pi(P, O)$ the trader profit when the first price is P and offered price is O.

When there is no price cap and all agents choose to enter the bubble

$$
\begin{aligned}
E(\pi(P,O)) &= \sum_{n=0}^{\infty}\sum_{j=1}^{3}\mathbb{P}(P=10^n, O=10^{n+j-1}) \times \pi(10^n, 10^{n+j-1}) \\
&= \mathbb{P}(P=10^0, O=10^0) \times \pi(10^0, 10^0) + \mathbb{P}(P=10^0, O=10^1) \times \pi(10^0, 10^1) \\
&+ \mathbb{P}(P=10^0, O=10^2) \times \pi(10^0, 10^2) \\
&+ \mathbb{P}(P=10^1, O=10^1) \times \pi(10^1, 10^1) + \mathbb{P}(P=10^1, O=10^2) \times \pi(10^1, 10^2) \\
&+ \mathbb{P}(P=10^1, O=10^3) \times \pi(10^1, 10^3) \\
&+ \mathbb{P}(P=10^2, O=10^2) \times \pi(10^2, 10^2) + \mathbb{P}(P=10^2, O=10^3) \times \pi(10^2, 10^3) \\
&+ \mathbb{P}(P=10^2, O=10^4) \times \pi(10^2, 10^4) \\
&+ \ldots \\
&= \mathbb{P}(P=10^0, O=10^0) \times (9 \times 10^0) + \mathbb{P}(P=10^0, O=10^1) \times (9 \times 10^1) \\
&+ \mathbb{P}(P=10^0, O=10^2) \times (-10^2) \\
&+ \mathbb{P}(P=10^1, O=10^1) \times (9 \times 10^1) + \mathbb{P}(P=10^1, O=10^2) \times (9 \times 10^2) \\
&+ \mathbb{P}(P=10^1, O=10^3) \times (-10^3) \\
&+ \mathbb{P}(P=10^2, O=10^2) \times (9 \times 10^2) + \mathbb{P}(P=10^2, O=10^3) \times (9 \times 10^3) \\
&+ \mathbb{P}(P=10^2, O=10^4) \times (-10^4) \\
&+ \ldots
\end{aligned}
$$

Conditioning first on the offered price (O), then on the first price (P)

$$
\begin{array}{rcl}
E(\pi(P,O)) & = & E\left(E(\pi(P,O))|O\right) \\
& = & \mathbb{P}(O=10^0)\left[\mathbb{P}(P=10^0|O=10^0)\times 9\times 10^0\right] \\
& + & \mathbb{P}(O=10^1)\left[\begin{array}{c} \mathbb{P}(P=10^0|O=10^1)\times 9\times 10^1 \\ +\mathbb{P}(P=10^1|O=10^1)\times 9\times 10^1 \end{array}\right] \\
& + & \mathbb{P}(O=10^2)\left[\begin{array}{c} \mathbb{P}(P=10^0|O=10^2)\times(-10^2) \\ +\mathbb{P}(P=10^1|O=10^2)\times 9\times 10^2 \\ +\mathbb{P}(P=10^2|O=10^2)\times 9\times 10^2 \end{array}\right] \\
& + & \mathbb{P}(O=10^3)\left[\begin{array}{c} \mathbb{P}(P=10^1|O=10^3)\times(-10^3) \\ +\mathbb{P}(P=10^2|O=10^3)\times 9\times 10^3 \\ +\mathbb{P}(P=10^3|O=10^3)\times 9\times 10^3 \end{array}\right] \\
& + & ... \\
& = & \frac{1}{2}\frac{1}{3}\times 9\times 10^0+(\frac{1}{2}+\frac{1}{4})\frac{1}{3}\times 9\times 10^1 \\
& + & \sum_{n=2}^{\infty}\left(\frac{1}{2^{n-1}}+\frac{1}{2^n}+\frac{1}{2^{n+1}}\right)\frac{1}{3}\left(\frac{4}{7}(-10^n)+\frac{3}{7}(9\times 10^n)\right) \\
& = & \frac{3}{2}+\frac{45}{2}+\frac{23}{6}\sum_{n=2}^{\infty}5^n \\
& = & +\infty
\end{array}
$$

Conditioning first on the first price (P), then on the offered price (O)

$$\begin{aligned}
E(\pi(P,O)) &= E\left(E(\pi(P,O))|P\right) \\
&= \mathbb{P}(P=10^0)\begin{bmatrix} \mathbb{P}(O=10^0|P=10^0)\times 9\times 10^0 \\ +\mathbb{P}(O=10^1|P=10^0)\times 9\times 10^1 \\ +\mathbb{P}(O=10^2|P=10^0)\times(-10^2) \end{bmatrix} \\
&+ \ \mathbb{P}(P=10^1)\begin{bmatrix} \mathbb{P}(O=10^1|P=10^1)\times 9\times 10^1 \\ +\mathbb{P}(P=10^2|O=10^1)\times 9\times 10^2 \\ +\mathbb{P}(P=10^3|O=10^1)\times(-10^3) \end{bmatrix} \\
&+ \ \mathbb{P}(O=10^2)\begin{bmatrix} \mathbb{P}(O=10^2|P=10^2)\times 9\times 10^2 \\ +\mathbb{P}(O=10^3|P=10^2)\times 9\times 10^3 \\ +\mathbb{P}(O=10^4|P=10^2)\times(-10^4) \end{bmatrix} \\
&+ \ ... \\
&= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\left(\frac{1}{3}\times 9\times 10^{n-1}+\frac{1}{3}\times 9\times 10^n-10^{n+1}\right) \\
&= -\frac{1}{6}\sum_{n=1}^{\infty}5^{n-1} \\
&= -\infty
\end{aligned}$$

Depending on the way conditioning is done, the ex-ante expected payoff is infinitely negative or positive. Such an ex-ante expected payoff is thus not well-defined.

When there is a price cap and all agents choose to enter the bubble In this case, using both ways of computing the expected profit give the same answer. Below, we compute the expected profit in the case where $K=1$.

Conditioning first on the offered price (O), then on the first price (P)

$$\begin{aligned}
E(\pi(P,O)) &= E\left(E(\pi(P,O))|O\right) \\
&= \frac{1}{3}(9+90-100) = -\frac{1}{3}
\end{aligned}$$

Conditioning first on the first price (P), then on the offered

price (O)

$$\begin{aligned} E(\pi(P,O)) &= E\left(E(\pi(P,O))|P\right) \\ &= \left(\frac{1}{3}\times 9+\frac{1}{3}\times 90-\frac{1}{3}\times 100\right)=-\frac{1}{3} \end{aligned}$$

Both ways of computing the conditional expectation yield the same conclusion: the ex-ante expected profit is negative.

3.2 Welfare analysis with risk aversion

We now analyze the welfare properties of our model when traders are risk averse. To simplify notations, we focus on the case in which $q=\frac{1}{2}$ and $m=10$. We show that, when the utility function is not bounded above, it is not possible to compute the ex-ante welfare of the players even if they are risk averse. The proof relies on the fact that the expected utility is well-defined if and only if the expected absolute utility is finite.

Consider that player i's utility function U_i is increasing and strictly concave with $U_i(+\infty)=+\infty$. We assume that players' initial wealth is null and that they can end up with negative wealth.

Case 1 : U_i is such that $U_i(0)=0$ Let us first assume that U_i is such that $U_i(0)=0$.

$$E\left(\left|U_i\left(W_f\right)\right|\right)=\frac{U_i(9)}{2}+\frac{U_i(9\times 10)}{4}+\sum_{n=2}^{n=+\infty}\left(\frac{1}{2}\right)^{n+1}\left(\frac{3}{7}U_i(9\times 10^n)+\frac{4}{7}\left|U_i(-10^n)\right|\right).$$

The IR condition imposes that for $n\geq 2$:

$$\frac{3}{7}U_i(9\times 10^n)+\frac{4}{7}U_i(-10^n)\geq U_i(0).$$

This yields:

$$E\left(\left|U_i\left(W_f\right)\right|\right)\geq\frac{U_i(9)}{2}+\frac{U_i(9\times 10)}{4}+\sum_{n=2}^{n=+\infty}\left(\frac{1}{2}\right)^{n+1}\left(\frac{4}{7}\left|U_i(-10^n)\right|-\frac{4}{7}U_i(-10^n)+U_i(0)\right).$$

$$E\left(\left|U_i\left(W_f\right)\right|\right)\geq\frac{U_i(9)}{2}+\frac{U_i(9\times 10)}{4}+\sum_{n=2}^{n=+\infty}\left(\frac{1}{2}\right)^{n+1}\left(-\frac{8}{7}U_i(-10^n)+U_i(0)\right).$$

By concavity of U_i, we have: for $x < 0$, $U_i(x) < U_i(0) + xU_i'(0)$. This yields for $x = 10^n$:

$$E\left(\left|U_i\left(W_f\right)\right|\right) > \frac{U_i(9)}{2} + \frac{U_i(9\times 10)}{4} + \sum_{n=2}^{n=+\infty}\left(\frac{1}{2}\right)^{n+1}\left(-\frac{8}{7}\left(U_i(0) - 10^n U_i'(0)\right) + U_i(0)\right)$$

$$E\left(\left|U_i\left(W_f\right)\right|\right) > \frac{U_i(9)}{2} + \frac{U_i(9\times 10)}{4} + \sum_{n=2}^{n=+\infty}\left(\frac{1}{2}\right)^{n+1}\left(-\frac{1}{7}U_i(0) + \frac{8}{7}10^n U_i'(0)\right).$$

Since the series $\sum_{n=2}^{n=+\infty}\left(\frac{1}{2}\right)^{n+1}\left(-\frac{1}{7}U_i(0) + \frac{8}{7}10^n U_i'(0)\right)$ does not converge, neither does $E\left(U_i\left(W_f\right)\right)$.

It is straightforward to extend this reasoning to the case in which for $U_i(0) \neq 0$ and in which $U_i(.)$ takes both positive and negative values. We now extend the proof to the cases in which $U_i(.)$ does not change sign.

Case 2 : U_i is such that $U_i(0) \neq 0$ Let us, for example, assume that for all w, $U_i(w) < 0$. This expected absolute utility is written as:

$$E\left(\left|U_i\left(W_f\right)\right|\right) = \frac{U_i(9)}{2} + \frac{U_i(9\times 10)}{4} + \sum_{n=2}^{n=+\infty}\left(\frac{1}{2}\right)^{n+1}\left(\frac{3}{7}\left|U_i(9\times 10^n)\right| + \frac{4}{7}\left|U_i(-10^n)\right|\right)$$

$$E\left(\left|U_i\left(W_f\right)\right|\right) = \frac{U_i(9)}{2} + \frac{U_i(9\times 10)}{4} + \sum_{n=2}^{n=+\infty}\left(\frac{1}{2}\right)^{n+1}\left(-\frac{3}{7}U_i(9\times 10^n) - \frac{4}{7}U_i(-10^n)\right).$$

Since $-\frac{3}{7}U_i(9\times 10^n) > 0$, we have :

$$E\left(\left|U_i\left(W_f\right)\right|\right) \geq \frac{U_i(9)}{2} + \frac{U_i(9\times 10)}{4} + \sum_{n=2}^{n=+\infty}\left(\frac{1}{2}\right)^{n+1}\left(-\frac{4}{7}U_i(-10^n)\right).$$

By concavity of U_i, we have: for $x < 0$, $U_i(x) < U_i(0) + xU_i'(0)$. This yields:

$$E\left(\left|U_i\left(W_f\right)\right|\right) \geq \frac{U_i(9)}{2} + \frac{U_i(9\times 10)}{4} + \sum_{n=2}^{n=+\infty}\left(\frac{1}{2}\right)^{n+1}\left(\frac{4}{7}\left(-U_i(0) + 10^n U_i'(0)\right)\right).$$

Again, since $\sum_{n=2}^{n=+\infty}\left(\frac{1}{2}\right)^{n+1}\left(\frac{4}{7}\left(-U_i(0) + 10^n U_i'(0)\right)\right)$ does not converge, the expected utility does not converge. For this proof, we only use the concavity of the utility function (the IR is not required).

4 The Subjective Quantal Response Equilibrium of Rogers, Palfrey, and Camerer, 2009

4.1 The general SQRE model

We derive the conditional probabilities to buy for risk-neutral traders observing prices of $P \in \{1, 10...\}$ in the SQRE model of Rogers, Palfrey, and Camerer (2009).

The main features of this model are as follows. First, as in the CH model, traders differ in their level of sophistication s. As in Camerer, Ho and Chong (2004), we assume that traders' types are distributed according to a Poisson distribution F. Let τ denote the average level of sophistication.

Second, each player s thinks that he understands the game differently than the others, and therefore form truncated beliefs about the fraction of h-level players according to $g_s(h) = \frac{f(h)}{\sum_{i=0}^{\max(s-\theta,0)} f(i)}$. The parameter $\theta \in N$ measures overconfidence. Camerer, Ho and Chong (2004) assume that people are overconfident and do not realize there are others using exactly as many thinking steps as they are, which implies that $\theta \geq 0$. The authors also assume that players doing $s \geq 1$ steps do not realize that others are using more than s steps of thinking, which would be plausible because the brain has limits (such as working memory in reasoning through complex games) and also does not always understand its own limits. We relax this assumption by considering that θ can be equal to 0.

Third, for reasons of parsimony and comparability to CH, we assume the error parameter θ is common to all traders, whatever their level of sophistication.

Fourth, as in the QRE model, players make mistakes about the others' type. The parameter $\lambda_{i,s}$ characterizes the responsiveness to expected payoffs of trader i. The following logistic specification of the stochastic choice function is assumed, so that, if the buy decision conditional on observing a price P for a level-s player of type i yields an expected profit of $u_{i,s}(B|P)$ while the no buy decision yields an expected profit of $u_\varnothing$, the probability to buy is:

$$\mathbb{P}_{i,s}(B|P) = \frac{e^{\lambda_{i,s} u_{i,s}(B|P)}}{e^{\lambda_{i,s} u_{i,s}(B|P)} + e^{\lambda_{i,s} u_\varnothing}}$$

Fifth, each level-s player is independently assigned by nature a response sensitivity, $\lambda_{i,s}$. For reasons of comparability both to CH and the QR, we assume that:

$$\lambda_{i,s} = \lambda_i + \gamma s,$$

where λ_i is drawn from a commonly known distribution, $F_i(\lambda_i)$. As in the HQRE model, we assume that the distribution $F_i(\lambda_i)$ is common knowledge, but traders' type, λ_i , is private information known only to i. We assume that F_i is uniform $[\Lambda - \frac{\epsilon}{2}, \Lambda + \frac{\epsilon}{2}]$. For computational reasons, we discretize this interval with a tick size t, therefore $f(\lambda_i) = \frac{1}{\frac{\epsilon}{t}+1} = f$.

Thus, it is a five parameter model with a Poisson parameter, τ , a spacing parameter, γ, an overconfidence parameter, θ, an average error parameter, Λ , and a parameter that controls the heterogeneity across traders' types, ϵ.

4.1.1 Cap K

Consider the environment in which there is a cap K on the initial price. For each price $P \in \{1, 10...100K\}$, each type i and each level s, we compute the player's expected utility if he buys, conditional on P, λ_i and s, in order to determine the theoretical probability to buy for each price as a function of the model's parameters.

Consider first the case of a trader observing a price $P = 100K$. This trader perfectly infers from this observation that he is third in the sequence. His expected payoff is he buys is thus $u_{i,s}(B|P = 100K) = 0$.

If he is a level-s player of type i, then $\lambda_{i,s} = \lambda_i + \gamma s$ and, he buys with probability:

$$\mathbb{P}_{i,s}(B|P = 100K) = \frac{1}{1 + e^{\lambda_i + \gamma s}}$$

Given the distribution of type-i players, the average probability to buy of a level-s player is:

$$\begin{aligned}
\mathbb{P}_s(B|P = 100K) &= \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \mathbb{P}_{i,s}(B|P = 100K)\mathbb{P}(\lambda_i = \lambda) \\
&= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1 + e^{\lambda+\gamma s}}
\end{aligned}$$

Finally, given that there is a fraction $f(s) = \frac{\tau^s \times \exp(-\tau)}{s!}$ of level-s traders in the population, the overall probability to buy conditional on the price being $P = 100K$ writes:

$$\mathbb{P}\left(B|P=100K\right) = \sum_{s=0}^{\infty} \frac{\tau^s \times \exp\left(-\tau\right)}{s!} f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda+\gamma s}}$$

Since we cannot compute this infinite sum, we numerically stop at $s_{\max} = 100$.

Notice that the CH model is a specific case of SQRE, with the following constraints on the parameters: $\epsilon = \Lambda = 0$, $\theta = 1$ and $\gamma \to \infty$. In this case indeed, the probability to buy when $P = 100K$ of a level-0 player is $\frac{1}{1+e^{\gamma\times 0}} = \frac{1}{2}$, while for level-s players with $s \geq 1$, $\lim_{\gamma\to\infty} \frac{1}{1+e^{\gamma\times s}} = 0$.

Notice also that the QR model is also a specific case of SQRE, with the following constraints on the parameters: $\tau = \gamma = \epsilon = 0$. In this case indeed, the population is only composed of homogeneous level-0 players, for which the probability to buy when $P = 100K$ is $\frac{1}{1+e^{\Lambda}}$.

Consider now the case of a trader observing a price $P < 100K$.

Let $q(K, P)$ be the probability not to be third conditional on observing the price P, when the price cap is K. For instance, when $K = 100$, $q(100, 1000) = q(100, 100) = \frac{1}{2}$ while $q(100, 10) = q(100, 1) = 1$, but when $K = 10,000$, $q(10000, 100000) = q(10000, 10000) = \frac{1}{2}$, $q(10000, 1000) = q(10000, 100) = \frac{3}{7}$ and $q(10000, 10) = q(10000, 1) = 1$.

The expected payoff of a level-s player of type i if he buys, $u_{i,s}$, depends first on his beliefs on the population (that is, on its truncation), and second, for each level s, on the average probability to buy of a level-s player observing a price $P' = 10P$. We have already defined above this probability for a level-s player observing a price $P' = 100K$, namely $P_s(B|P' = 100K)$. This will enable us to find the expected payoff of a level-s player of type i observing $P = 10K$ if he buys, $u_{i,s}(B|P = 10K)$, thus the probability with which a level-s player observing $P = 10K$ buys. Recursively, we can therefore find the probability with which a level-s player buys when he observes $P = K, P = K/10$, and so on.

- If he is a level-s player, with $s - \theta \leq 0$, he thinks that all traders observing $P' = 10P$ are level-0 players who buy with an average probability $\mathbb{P}_{s=0}(P' = 10P)$. His expected payoff if he buys is therefore:

$$u_{i,s\leq\theta}(B|P) = 10q(K, P) \times \mathbb{P}_{s=0}(B|P' = 10P)$$

Consequently:

$$\mathbb{P}_{i,s\leq\theta}(B|P) = \frac{e^{(\lambda_i+\gamma s)u_{i,s\leq\theta}(B|P)}}{e^{(\lambda_i+\gamma s)u_{i,s\leq\theta}(B|P)} + e^{(\lambda_i+\gamma s)}}$$

Given the distribution of type-i players, the average probability to buy of level-s players is:

$$\begin{aligned}\mathbb{P}_{s\leq\theta}(B|P) &= \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \mathbb{P}_{i,s\leq\theta}(B|P)\mathbb{P}(\lambda_i=\lambda) \\ &= f\sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{(\lambda_i+\gamma s)u_{i,s\leq\theta}(B|P)}}{e^{(\lambda_i+\gamma s)u_{i,s\leq\theta}(B|P)} + e^{(\lambda_i+\gamma s)}}\end{aligned}$$

- If he is a level-s player, with $s-\theta>0$, he thinks that the next player observing the price $P_3 = 10\times P_2$ is a mixture of level-0, ..., level j,...,level $s-\theta$. Consequently, his expected profit if he buys writes:

$$u_{i,s>\theta}(B|P) = 10q(10K,P)\times\frac{\sum_{j=0}^{s-\theta}\mathbb{P}_{s=j}(B|P'=10P)f(j)}{\sum_{j=0}^{s-\theta}f(j)}$$

where $f(j) = e^{-\tau}\frac{\tau^j}{j!}$, while his profit if he does not buy is $u_\varnothing = 1$.
His probability to buy is therefore:

$$\mathbb{P}_{i,s>\theta}(B|P) = \frac{e^{(\lambda_i+\gamma s)\times u_{i,s>\theta}(B|P)}}{e^{(\lambda_i+\gamma s)\times u_{i,s>\theta}(B|P)} + e^{\lambda_i+\gamma s}}$$

Given the distribution of players of type i,

$$\begin{aligned}\mathbb{P}_{s>\theta}(B|P) &= \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \mathbb{P}_{i,s>\theta}(B|P)\mathbb{P}(\lambda_i=\lambda) \\ &= f\sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{(\lambda_i+\gamma s)\times u_{i,s>\theta}(B|P)}}{e^{(\lambda_i+\gamma s)\times u_{i,s>\theta}(B|P)} + e^{\lambda_i+\gamma s}}\end{aligned}$$

Finally, given that there is a fraction $f(s) = \frac{\tau^s\times\exp(-\tau)}{s!}$ of level-s traders in the population, the overall probability to buy conditional on the price being $P < 100K$ writes:

$$\begin{aligned}\mathbb{P}\left(B|P\right) &= \sum_{s=0}^{\max(s-\theta,0)} \frac{\tau^s \times \exp\left(-\tau\right)}{s!} f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{(\lambda_i+\gamma s)u_{i,s\leq\theta}(B|P)}}{e^{(\lambda_i+\gamma s)u_{i,s\leq\theta}(B|P)}+e^{(\lambda_i+\gamma s)}} \\ &+ \sum_{s=\max(s-\theta,0)+1}^{\infty} \frac{\tau^s \times \exp\left(-\tau\right)}{s!} f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{(\lambda_i+\gamma s)\times u_{i,s>\theta}(B|P)}}{e^{(\lambda_i+\gamma s)\times u_{i,s>\theta}(B|P)}+e^{\lambda_i+\gamma s}}\end{aligned}$$

4.1.2 No cap

Consider now the environment in which there is no cap on the initial price. Consider first the case of a trader observing a price $P \geq 100$. Conditional on observing this price, traders have a probability $\frac{3}{7}$ not to be third.

- If he is a level-0 player of type i, then $\lambda_{i,s} = \lambda_i$. Given that $\theta \geq 0$, he thinks that all traders observing $P' = 10P$ are level-0 players who buy with an average probability $\mathbb{P}_{s=0}(P' \geq 100)$. His expected payoff if he buys is therefore:

$$u_{i,s=0}(B|P \geq 100) = 10 \times \frac{3}{7} \times \mathbb{P}_{s=0}(B|P' \geq 100)$$

Consequently:

$$\mathbb{P}_{i,s=0}(B|P \geq 100) = \frac{e^{\lambda_i u_{i,s=0}(B|P\geq 100)}}{e^{\lambda_i u_{i,s=0}(B|P\geq 100)}+e^{\lambda_i}}$$

Given the distribution of players of type i,

$$\begin{aligned}\mathbb{P}_{s=0}(B|P \quad \geq \quad 100) &= \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \mathbb{P}_{i,s=0}(B|P \geq 100)\mathbb{P}(\lambda_i = \lambda) \\ &= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{\lambda_i \frac{30}{7}\mathbb{P}_{s=0}(B|P'\geq 100)}}{e^{\lambda_i \frac{30}{7}\mathbb{P}_{s=0}(B|P'\geq 100)}+e^{\lambda_i}}\end{aligned}$$

Therefore, $\mathbb{P}_{s=0}(B|P \geq 100)$ is a fixed point, solution of the equation above.

- If he is a level-s player, with $0 < s \leq \theta$, then $\lambda_{i,s} = \lambda_i + \gamma s$ and he thinks that all traders observing $P' = 10P$ are level-0 players who buy with an average probability $\mathbb{P}_{s=0}(P' \geq 100)$. His expected payoff if he buys is therefore:

$$u_{i,s\leq\theta}(B|P) = 10 \times \frac{3}{7} \times \mathbb{P}_{s=0}(B|P' = 10P)$$

Consequently:

$$\mathbb{P}_{i,s\leq\theta}(B|P) = \frac{e^{(\lambda_i+\gamma s)u_{i,s\leq\theta}(B|P)}}{e^{(\lambda_i+\gamma s)u_{i,s\leq\theta}(B|P)} + e^{(\lambda_i+\gamma s)}}$$

Given the distribution of players of type i,

$$\begin{aligned}
\mathbb{P}_{s\leq\theta}(B|P \geq 100) &= \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \mathbb{P}_{i,s\leq\theta}(B|P \geq 100)\mathbb{P}(\lambda_i = \lambda) \\
&= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{(\lambda_i+\gamma s)\frac{30}{7}\mathbb{P}_{s=0}(B|P'\geq 100)}}{e^{(\lambda_i+\gamma s)\frac{30}{7}\mathbb{P}_{s=0}(B|P'\geq 100)} + e^{(\lambda_i+\gamma s)}}
\end{aligned}$$

- If he is a level-s player, with $s > \theta$, then $\lambda_{i,s} = \lambda_i + \gamma s$ and he thinks that the traders observing $P' = 10P$ are a mixture of level-0,..., level-j,..., level $s - \theta$ players who buy with an average probability $\mathbb{P}_j(B|P' \geq 100)$. His expected payoff if he buys is therefore:

$$u_{i,s>\theta}(B|P \geq 100) = 10 \times \frac{3}{7} \times \frac{\sum_{j=0}^{s-\theta} \mathbb{P}_j(B|P' \geq 100)f(j)}{\sum_{j=0}^{s-\theta} f(j)}$$

We have computed above $\mathbb{P}_j(B|P' \geq 100)$ for $j \in \{0, ..., \theta\}$. For $\theta < s \leq 2\theta$,

$$u_{i,s>\theta}(B|P \geq 100) = 10 \times \frac{3}{7} \times \frac{\sum_{j=0}^{s-\theta} \mathbb{P}_{j\leq\theta}(B|P' \geq 100)f(j)}{\sum_{j=0}^{s-\theta} f(j)}.$$

Consequently, the expected utility of buying for level-s players, for $\theta < s \leq 2\theta$, is well-defined. Then for $s > 2\theta$,

$$\begin{aligned}
u_{i,s>\theta}(B|P \geq 100) &= 10 \times \frac{3}{7} \\
&\times \frac{\sum_{j=0}^{\theta} \mathbb{P}_{j\leq\theta}(B|P' \geq 100)f(j) + \sum_{j=\theta+1}^{s-\theta} \mathbb{P}_{j>\theta}(B|P' \geq 100)f(j)}{\sum_{j=0}^{s-\theta} f(j)}
\end{aligned}$$

When $\theta > 0$, the expected utility of buying for level-s players, for $s > 2\theta$, can be defined recursively (by recurrence).

$$\mathbb{P}_{i,s>\theta}(B|P \geq 100) = \frac{e^{(\lambda_i+\gamma s)u_{i,s>\theta}(B|P\geq 100)}}{e^{(\lambda_i+\gamma s)u_{i,s>\theta}(B|P\geq 100)} + e^{(\lambda_i+\gamma s)}}$$

Given the distribution of players of type i,

$$\begin{aligned}
\mathbb{P}_{s>\theta}(B|P \geq 100) &= \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \mathbb{P}_{i,s>\theta}(B|P \geq 100)\mathbb{P}(\lambda_i = \lambda) \\
&= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{(\lambda_i+\gamma s)u_{i,s>\theta}(B|P\geq 100)}}{e^{(\lambda_i+\gamma s)u_{i,s>\theta}(B|P\geq 100)} + e^{(\lambda_i+\gamma s)}}
\end{aligned}$$

When $\theta = 0$ however, the probability with which a level-s player buys is a fixed point. Indeed, if $p_j = \mathbb{P}_j(B|P' \geq 100)$,

$$u_{i,s>0}(B|P \geq 100) = 10 \times \frac{3}{7} \times \frac{\sum_{j=0}^{s-1} p_j f(j) + p_s f(s)}{\sum_{j=0}^{s} f(j)}$$

Probabilities p_j for $j < s$ can be found recursively. Given the distribution of traders' types:

$$\mathbb{P}_s(B|P \geq 100) = p_s = f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{(\lambda+\gamma s)\frac{30}{7}\frac{\sum_{j=0}^{s-1} p_j f(j)+p_s f(s)}{\sum_{j=0}^{s} f(j)}}}{e^{(\lambda+\gamma s)\frac{30}{7}\frac{\sum_{j=0}^{s-1} p_j f(j)+p_s f(s)}{\sum_{j=0}^{s} f(j)}} + e^{(\lambda+\gamma s)}}$$

Finally, given that there is a fraction $f(s) = \frac{\tau^s \times \exp(-\tau)}{s!}$ of level-s traders in the population, the overall probability to buy conditional on the price being $P \geq 100$ writes:

$$\begin{aligned}
\mathbb{P}(B|P) &= \sum_{s=0}^{\max(s-\theta,0)} e^{-\tau}\frac{\tau^s}{s!} f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{(\lambda_i+\gamma s)u_{i,s\leq\theta}(B|P)}}{e^{(\lambda_i+\gamma s)u_{i,s\leq\theta}(B|P)} + e^{(\lambda_i+\gamma s)}} \\
&+ \sum_{s=\max(s-\theta,0)+1}^{\infty} e^{-\tau}\frac{\tau^s}{s!} f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{(\lambda_i+\gamma s)\times u_{i,s>\theta}(B|P)}}{e^{(\lambda_i+\gamma s)\times u_{i,s>\theta}(B|P)} + e^{\lambda_i+\gamma s}}
\end{aligned}$$

Consider now the case of a trader observing a price $P < 100$. Conditional on observing this price, traders have a probability 1 not to be third. The probability to buy of a level-s player of type i can be found recursively as in the case where there is a cap.

4.2 The TQRE model of Rogers, Palfrey, and Camerer, 2009, and its limit to the Cognitive Hierarchy model

In this subsection, we show that the TQRE model is a specific case of the SQRE model, with the constraints $\varepsilon = \Lambda = 0$, and $\theta = 1$.

Indeed, the SQRE constrained on ε, Λ and θ is a two-parameter model with Poisson parameter, τ , and a spacing parameter, γ. First, as in the CH model, traders differ in their level of sophistication s. Traders' levels are distributed according to a Poisson distribution F with mean τ. When $\theta = 1$, each player s thinks that he understands the game differently than the others, and therefore form truncated beliefs about the fraction of h-level players according to $g_s(h) = \frac{f(h)}{\sum_{i=0}^{s-1} f(i)}$. Second, as in the QRE model, players make mistakes about the others' type. In the SQRE, the parameter $\lambda_{i,s}$ characterizes the responsiveness to expected payoffs. The following logistic specification of the stochastic choice function is assumed, so that, if the buy decision conditional on observing a price P yields an expected profit of $u(B|P)$ while the no buy decision yields an expected profit of $u_{\varnothing}$, the probability to buy is:

$$\mathbb{P}(B|P) = \frac{e^{\lambda_{i,s} u(B|P)}}{e^{\lambda_{i,s} u(B|P)} + e^{\lambda_{i,s} u_{\varnothing}}}.$$

When $\varepsilon = \Lambda = 0$, there is no heterogeneity across traders' types, as $\lambda_i = 0$, but skill levels are Poisson distributed and equally space $\lambda_s = \gamma \times s$. This model therefore corresponds to the discretized TQRE model of Rogers, Palfrey, and Camerer, 2009.

Below, we derive the probabilities to buy conditional on each price, as a function of τ and γ.

4.2.1 Cap K=1

Consider the environment in which there is a cap $K = 1$ on the initial price. We derive the conditional probabilities to buy for risk-neutral traders observing prices of $P \in \{1, 10, 100\}$ in the SQRE model of Rogers, Palfrey, and Camerer (2009), constrained to $\varepsilon = \Lambda = 0$, and $\theta = 1$. Given that there is no heterogeneity in λ_i, it is always the case that $\mathbb{P}_s(B|P) = \mathbb{P}_{i,s}(B|P)$.

Consider first the case of a trader observing a price $P = 100$. This trader perfectly infers from this observation that he is third in the sequence. His expected payoff is he buys is thus $u_{i,s}(B|P = 100) = 0$.

If he is a level-s player, then $\lambda_{i,s} = \gamma s$ and, he buys with probability:

$$\mathbb{P}_s(B|P = 100) = \frac{1}{1 + e^{\gamma s}}$$

As in the CH model, notice that level-0 players buy with probability $\frac{1}{2}$. In contrast though, when γ is finite, higher-level players also buy in the TQRE model, but the probability with which they make a mistake decreases with their level of sophistication and with γ. When $\gamma \to \infty$, no player with $s > 0$ buys, and the limit of the TQRE model is thus the CH model.

Finally, given that there is a fraction $f(s) = \frac{\tau^s \times \exp(-\tau)}{s!}$ of level-s traders in the population, the overall probability to buy conditional on the price being $P = 100K$ writes:

$$\mathbb{P}(B|P = 100) = \sum_{s=0}^{\infty} \frac{\tau^s \times \exp(-\tau)}{s!} \frac{1}{1 + e^{\gamma s}}$$

Since we cannot compute this infinite sum, we numerically stop at $s_{\max} = 100$.

Consider now the case of a trader observing a price $P = 10$. This trader perfectly infers from this observation that he is second in the sequence. The expected payoff of a level-s player if he buys, u_s, depends first on his beliefs on the population (that is, on its truncation), and second, for each level s, on the average probability to buy of a level-s player observing a price $P' = 10P = 100$. We have already defined above this probability for a level-s player observing a price $P' = 100$, namely $\mathbb{P}_s(B|P' = 100)$. This will enable us to find the expected payoff of a level-s player of type i observing $P = 10$ if he buys, $u_s(B|P = 10)$, thus the probability with which a level-s player observing $P = 10$ buys.

- If he is a level-s player, with $s \leq 1$, he thinks that all traders observing $P' = 100$ are level-0 players who buy with an average probability $\mathbb{P}_{s=0}(P' = 100) = \frac{1}{2}$. His expected payoff if he buys is therefore:

$$u_{s\leq 1}(B|P = 10) = 10 \times \frac{1}{2}$$

Consequently:

$$\mathbb{P}_{s\leq 1}(B|P = 10) = \frac{e^{5\gamma s}}{e^{5\gamma s} + e^{\gamma s}}$$

Again, notice that level-0 players buy with probability $\frac{1}{2}$ as in the CH model. Now, level-1 players buy with a larger probability when they observe $P = 10$ than when they observe $P = 100$, since $\frac{e^{5\gamma}}{e^{5\gamma}+e^{\gamma}} > \frac{1}{1+e^{\gamma}}$. In the CH model, which is obtained at the limit when $\gamma \to \infty$, level-1 traders even buy with probability 1, as they think that the population is only composed of level-0 traders who buy with probability $\frac{1}{2}$.

- If he is a level-s player, with $s > 1$, he thinks that the next player observing the price $P_3 = 10 \times P_2 = 100$ is a mixture of level-0, ..., level-j,..., level $s - 1$. Consequently, his expected profit if he buys writes:

$$\begin{aligned} u_{s>1}(B|P = 10) &= 10 \times \frac{\sum_{k=0}^{s-1} \mathbb{P}_{s=k}(B|P' = 100) f(k)}{\sum_{k=0}^{s-1} f(k)} \\ &= 10 \times \frac{\sum_{j=0}^{s-1} \frac{1}{1+e^{\gamma j}} f(j)}{\sum_{j=0}^{s-1} f(j)} \end{aligned}$$

where $f(j) = e^{-\tau}\frac{\tau^j}{j!}$, while his profit if he does not buy is $u_{\varnothing} = 1$.
His probability to buy is therefore:

$$\mathbb{P}_{s>1}(B|P = 10) = \frac{e^{10\gamma s \frac{\sum_{j=0}^{s-1} \frac{1}{1+e^{\gamma j}} \frac{\tau^j}{j!}}{\sum_{j=0}^{s-1} \frac{\tau^j}{j!}}}}{e^{10\gamma s \frac{\sum_{j=0}^{s-1} \frac{1}{1+e^{\gamma j}} \frac{\tau^j}{j!}}{\sum_{j=0}^{s-1} \frac{\tau^j}{j!}}} + e^{\gamma s}}$$

In the CH model, which is obtained at the limit when $\gamma \to \infty$, only level-0 players buy when they observe $P = 100$. We have seen that this induces level-1 players to buy when $P = 10$. What about higher level players? If s and τ are such that:

$$10 \frac{\sum_{j=0}^{s-1} \frac{1}{1+e^{\gamma j}} \frac{\tau^j}{j!}}{\sum_{j=0}^{s-1} \frac{\tau^j}{j!}} > 1,$$

then $\mathbb{P}_{s>1}(B|P=10) \to 1$, else $\mathbb{P}_{s>1}(B|P=10) \to 0$. For a fixed τ, there may exist low level-s players for which the probability with which the trader observing $P=100$ would be sufficiently large to induce him to buy, given that they have a truncated belief on the population, while for higher level players, who have a more accurate perception of the proportion of level-0 players, this would not be the case. The number of levels s such that buying is profitable decreases with τ, as this parameter influences the "true" proportion of level-0 players in the population.

Finally, given that there is a fraction $f(s) = \frac{\tau^s \times \exp(-\tau)}{s!}$ of level-s traders in the population, the overall probability to buy conditional on the price being $P=10$ writes:

$$\mathbb{P}(B|P=10) = \frac{1}{2}e^{-\tau} + \sum_{s=1}^{\infty} e^{-\tau}\frac{\tau^s}{s!}\frac{e^{10\gamma s \frac{\sum_{j=0}^{s-1}\frac{1}{1+e^{\gamma j}}\frac{\tau^j}{j!}}{\sum_{j=0}^{s-1}\frac{\tau^j}{j!}}}}{e^{10\gamma s \frac{\sum_{j=0}^{s-1}\frac{1}{1+e^{\gamma j}}\frac{\tau^j}{j!}}{\sum_{j=0}^{s-1}\frac{\tau^j}{j!}}} + e^{\gamma s}}$$

Consider finally the case of a trader observing a price $P=1$. This trader perfectly infers from this observation that he is first in the sequence. The expected payoff of a level-s player of type i if he buys, u_s, depends first on his beliefs on the population (that is, on its truncation), and second, for each level s, on the average probability to buy of a level-s player observing a price $P' = 10P = 10$. We have already defined above this probability for a level-s player observing a price $P' = 10$, namely $\mathbb{P}_s(B|P'=10)$. This will enable us to find the expected payoff of a level-s player of type i observing $P=1$ if he buys, $u_s(B|P=1)$, thus the probability with which a level-s player observing $P=1$ buys.

- If he is a level-s player, with $s \leq 1$, he thinks that all traders observing $P' = 10$ are level-0 players who buy with an average probability $\mathbb{P}_{s=0}(P' = 10) = \frac{1}{2}$. His expected payoff if he buys is therefore:

$$u_{s\leq 1}(B|P=1) = 10 \times \frac{1}{2}$$

Consequently:

$$\mathbb{P}_{s\leq 1}(B|P=1) = \frac{e^{5\gamma s}}{e^{5\gamma s} + e^{\gamma s}}$$

Again, notice that level-0 players buy with probability $\frac{1}{2}$ as in the CH model. Level-1 players buy as often as when they observe $P = 10$.

- If he is a level-s player, with $s > 1$, he thinks that the next player observing the price $P_2 = 10 \times P_1 = 10$ is a mixture of level-0, ..., level $s-1$. Consequently, his expected profit if he buys writes:

$$
\begin{aligned}
u_{s>1}(B|P=1) &= 10 \times \frac{\sum_{j=0}^{s-1} \mathbb{P}_{s=j}(B|P'=10) f(j)}{\sum_{j=0}^{s-1} f(j)} \\
&= 10 \times \frac{\frac{1}{2} + \sum_{j=1}^{s-1} \frac{\tau^j}{j!} \frac{e^{10\gamma j \frac{\sum_{k=0}^{j-1} \frac{1}{1+e^{\gamma k}} \frac{\tau^k}{k!}}{\sum_{k=0}^{j-1} \frac{\tau^k}{k!}}}}{e^{10\gamma j \frac{\sum_{k=0}^{j-1} \frac{1}{1+e^{\gamma k}} \frac{\tau^k}{k!}}{\sum_{k=0}^{j-1} \frac{\tau^k}{k!}}} + e^{\gamma j}}}{\sum_{j=0}^{s-1} \frac{\tau^j}{j!}}
\end{aligned}
$$

The probability with which a level-s player buys for $s > 1$ is:

$$
\mathbb{P}_{s>1}(B|P=1) = \frac{e^{10\gamma s \frac{\frac{1}{2} + \sum_{j=1}^{s-1} \frac{\tau^j}{j!} \frac{e^{10\gamma j \frac{\sum_{k=0}^{j-1} \frac{1}{1+e^{\gamma k}} \frac{\tau^k}{k!}}{\sum_{k=0}^{j-1} \frac{\tau^k}{k!}}}}{e^{10\gamma j \frac{\sum_{k=0}^{j-1} \frac{1}{1+e^{\gamma k}} \frac{\tau^k}{k!}}{\sum_{k=0}^{j-1} \frac{\tau^k}{k!}}} + e^{\gamma j}}}{\sum_{j=0}^{s-1} \frac{\tau^j}{j!}}}}{e^{10\gamma s \frac{\frac{1}{2} + \sum_{j=1}^{s-1} \frac{\tau^j}{j!} \frac{e^{10\gamma j \frac{\sum_{k=0}^{j-1} \frac{1}{1+e^{\gamma k}} \frac{\tau^k}{k!}}{\sum_{k=0}^{j-1} \frac{\tau^k}{k!}}}}{e^{10\gamma j \frac{\sum_{k=0}^{j-1} \frac{1}{1+e^{\gamma k}} \frac{\tau^k}{k!}}{\sum_{k=0}^{j-1} \frac{\tau^k}{k!}}} + e^{\gamma j}}}{\sum_{j=0}^{s-1} \frac{\tau^j}{j!}}} + e^{\gamma s}}
$$

In the CH model, which is obtained at the limit when $\gamma \to \infty$, we have seen that higher level players may buy when observing $P = 10$, depending on the value of τ. Let us assume that in the constrained model, τ is such that level-s players buy when $P = 10$ if $s \leq \bar{S}$, and do not buy if $s > \bar{S}$, with $\bar{S} \geq 1$. Given the expected utility of a level-s player observing $P = 10$ if he buys, $\bar{S}$ is such that:

$$
10 \frac{\frac{1}{2}}{\sum_{j=0}^{\bar{S}-1} \frac{\tau^j}{j!}} > 1,
$$

while

$$10\frac{\frac{1}{2}}{\sum_{j=0}^{\bar{S}}\frac{\tau^j}{j!}} < 1.$$

Would a level-$\bar{S}+1$ player buy when $P = 1$? When $\gamma \to \infty$, his expected utility if he buys writes:

$$u_{i,s=\bar{S}+1}(B|P=1) = 10 \times \frac{\sum_{j=0}^{\bar{S}} \mathbb{P}_{s=j}(B|P'=10)f(j)}{\sum_{j=0}^{\bar{S}} f(j)}$$

But when $P = 10$, level-0 player buy with probability $\frac{1}{2}$ while all players such that $s \leq \bar{S}$ buy with probability 1. Consequently,

$$u_{i,s} = 10 \times \frac{\frac{1}{2} + \sum_{j=1}^{\bar{S}}\frac{\tau^j}{j!}}{\sum_{j=0}^{\bar{S}}\frac{\tau^j}{j!}} = 10 - 10\frac{\frac{1}{2}}{\sum_{j=0}^{\bar{S}}\frac{\tau^j}{j!}},$$

which is strictly larger than 1, the expected utility of the trader if he does not buy, by definition of $\bar{S}$ above. Consequently, even higher level players are induced to buy when $P = 1$: there is a snowballing effect.

Finally, given that there is a fraction $f(s) = \frac{\tau^s \times \exp(-\tau)}{s!}$ of level-s traders in the population, the overall probability to buy conditional on the price being $P = 1$ writes:

$$\begin{aligned}
\mathbb{P}(B|P=10) &= (\frac{1}{2} + \tau\frac{e^{5\gamma}}{e^{5\gamma}+e^{\gamma}})e^{-\tau} \\
&+ \sum_{s=2}^{\infty} e^{-\tau}\frac{\tau^s}{s!}\frac{e^{10\gamma s\frac{\frac{1}{2}+\sum_{j=1}^{s-1}\frac{\tau^j}{j!}\frac{e^{10\gamma j\frac{\sum_{k=0}^{j-1}\frac{1}{1+e^{\gamma k}}\frac{\tau^k}{k!}}{\sum_{k=0}^{j-1}\frac{\tau^k}{k!}}}}{e^{10\gamma j\frac{\sum_{k=0}^{j-1}\frac{1}{1+e^{\gamma k}}\frac{\tau^k}{k!}}{\sum_{k=0}^{j-1}\frac{\tau^k}{k!}}}+e^{\gamma j}}}{\sum_{j=0}^{s-1}\frac{\tau^j}{j!}}}}{e^{10\gamma s\frac{\frac{1}{2}+\sum_{j=1}^{s-1}\frac{\tau^j}{j!}\frac{e^{10\gamma j\frac{\sum_{k=0}^{j-1}\frac{1}{1+e^{\gamma k}}\frac{\tau^k}{k!}}{\sum_{k=0}^{j-1}\frac{\tau^k}{k!}}}}{e^{10\gamma j\frac{\sum_{k=0}^{j-1}\frac{1}{1+e^{\gamma k}}\frac{\tau^k}{k!}}{\sum_{k=0}^{j-1}\frac{\tau^k}{k!}}}+e^{\gamma j}}}{\sum_{j=0}^{s-1}\frac{\tau^j}{j!}}}+e^{\gamma s}}
\end{aligned}$$

4.2.2 No cap

Consider the environment in which there is no cap on the initial price. We derive the conditional probabilities to buy for risk-neutral traders observing prices of $P \in \{1, 10, 100, ...\}$ in the SQRE model of Rogers, Palfrey, and Camerer (2009), constrained to $\varepsilon = \Lambda = 0$, and $\theta = 1$.
Consider first the case of a trader observing a price $P \geq 100$. Conditional on observing this price, traders have a probability $\frac{3}{7}$ not to be third.

- If he is a level-0 player, then $\lambda_{i,s} = 0$ and buys with probability:

$$\mathbb{P}_0(B|P \geq 100) = \frac{e^0}{e^0 + e^0} = \frac{1}{2}$$

- If he is a level-1 player, then $\lambda_{i,s} = \gamma$ and he thinks that all traders observing $P' = 10P$ are level-0 players who buy with an average probability $\mathbb{P}_{s=0}(P' \geq 100)$. Given that we have computed this probability above, his expected payoff if he buys is therefore:

$$u_1(B|P \geq 100) = 10 \times \frac{3}{7} \times \mathbb{P}_{s=0}(B|P' \geq 100) = \frac{15}{7}$$

Consequently:

$$\mathbb{P}_1(B|P \geq 100) = \frac{e^{\frac{15}{7}\gamma}}{e^{\frac{15}{7}\gamma} + e^{\gamma}}$$

- If he is a level-s player, with $s > 1$, then $\lambda_{i,s} = \gamma s$ and he thinks that the traders observing $P' = 10P$ are a mixture of level-0,...,level j,.., level $s-1$ players who buy with an average probability $\mathbb{P}_j(P' \geq 100)$. His expected payoff if he buys is therefore:

$$u_{s>1}(B|P \geq 100) = 10 \times \frac{3}{7} \times \frac{\sum_{j=0}^{s-1} \mathbb{P}_j(B|P' \geq 100) f(j)}{\sum_{j=0}^{s-1} f(j)}$$

We have computed above $\mathbb{P}_j(B|P' \geq 100)$ for $j \in \{0, 1\}$. The expected utility of buying for level-2 players is thus well-defined:

$$u_2(B|P \geq 100) = 10 \times \frac{3}{7} \times \frac{\frac{1}{2} + \tau \frac{e^{\frac{15}{7}\gamma}}{e^{\frac{15}{7}\gamma} + e^{\gamma}}}{1 + \tau}$$

This enables us to compute $\mathbb{P}_2(B|P \geq 100)$, which it itself used to find the expected utility of buying for level-3 players. Finally, the probability to buy of a level-s player observing $P \geq 100$ can be computed recursively.

In the CH model, which is obtained at the limit when $\gamma \to \infty$, level-0 players buy with probability $\frac{1}{2}$ when $P \geq 100$, thus level-1 players buy since $\frac{15}{7} > 1$, thus level-2 players buy since $10\frac{3}{7}\frac{\frac{1}{2}+\tau}{1+\tau} > 1$ whatever τ. Finally, all level-s players buy for $s \geq 1$ since whatever τ:

$$\frac{30}{7}\frac{\frac{1}{2}+\sum_{j=1}^{s-1}\frac{\tau^j}{j!}}{\sum_{j=0}^{s-1}\frac{\tau^j}{j!}} > 1$$

Finally, given that there is a fraction $f(s) = \frac{\tau^s \times \exp(-\tau)}{s!}$ of level-s traders in the population, the overall probability to buy conditional on the price being $P \geq 100$ writes:

$$\mathbb{P}\left(B|P \geq 100\right) = \sum_{s=0}^{\infty} e^{-\tau}\frac{\tau^s}{s!}\mathbb{P}_s(B|P \geq 100)$$

Consider now the case of a trader observing a price $P < 100$. Conditional on observing this price, traders have a probability 1 not to be third. The probability to buy of a level-s player of type i can be found as in the case where there is a cap.

In the CH model, which is obtained at the limit when $\gamma \to \infty$, as all level-s players where $s \geq 1$ buy when $P \geq 100$, all level-s players where $s \geq 1$ buy when $P < 100$, and only level-0 players buy with a probability $\frac{1}{2}$. Consequently, the probability to buy in the CH model is constant whatever the price and equal to $1 - \frac{1}{2}e^{-\tau}$.

4.3 The HQRE model of Rogers, Palfrey, and Camerer, 2009, and its limit to the Quantal Response Equilibrium model

In this subsection, we show that the HQRE model is a specific case of the SQRE model, with the constraint $\tau = 0$ (γ and $\theta = 1$ do not play a role in this case).

Indeed, the SQRE constrained to $\tau = 0$ is a two-parameter model with an average error parameter, Λ , and a parameter that controls the heterogeneity across traders' types, ϵ. First, as in the QRE model, players make mistakes about the others' type. In the SQRE, the parameter $\lambda_{i,s}$ characterizes the responsiveness to expected payoffs of trader i. The following logistic specification of the stochastic choice function is assumed, so that, if the buy decision conditional on observing a price P yields an expected profit of $u(B|P)$ while the no buy decision yields an expected profit of $u_{\varnothing}$, the probability to buy is:

$$\mathbb{P}_i(B|P) = \frac{e^{\lambda_{i,s} u(B|P)}}{e^{\lambda_{i,s} u(B|P)} + e^{\lambda_{i,s} u_{\varnothing}}},$$

Second, when $\tau = 0$, all players are level-0 players, therefore $\lambda_{i,s} = \lambda_i$, where λ_i is drawn from a commonly known distribution, $F_i(\lambda_i)$, which is uniform on $[\Lambda - \frac{\epsilon}{2}, \Lambda + \frac{\epsilon}{2}]$. We discretize this interval with a tick size t, therefore $f(\lambda_i) = \frac{1}{\frac{\epsilon}{t}+1} = f$.

Below, we derive the probabilities to buy conditional on each price, as a function of Λ and ϵ.

4.3.1 Cap K=1

Consider the environment in which there is a cap $K = 1$ on the initial price. We derive the conditional probabilities to buy for risk-neutral traders observing prices of $P \in \{1, 10, 100\}$ in the SQRE model of Rogers, Palfrey, and Camerer (2009), constrained to $\tau = 0$.

Consider first the case of a trader observing a price $P = 100$. This trader perfectly infers from this observation that he is third in the sequence. His expected payoff is he buys is thus $u_{i,s}(B|P = 100) = 0$.

If he is a type i, then $\lambda_{i,s} = \lambda_i$ and he buys with probability:

$$\mathbb{P}_i(B|P = 100) = \frac{1}{1 + e^{\lambda_i}}$$

Given the distribution of type-i players, the average probability to buy at price $P = 100$ is:

$$\begin{aligned}
\mathbb{P}(B|P = 100) &= \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \mathbb{P}_i(B|P = 100)\mathbb{P}(\lambda_i = \lambda) \\
&= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1 + e^{\lambda}}
\end{aligned}$$

Recall that the QR model is also a specific case of SQRE, with the additional constraint $\epsilon = 0$. In this case, the probability to buy when $P = 100$ simplifies to $\frac{1}{1+e^{\Lambda}}$.

Consider now the case of a trader observing a price $P = 10$. This trader perfectly infers from this observation that he is second in the sequence. The expected payoff of a player of type i if he buys, u_i, depends on the average probability to buy of players observing a price $P' = 10P = 100$. We have already defined this probability above, namely $P(B|P' = 100)$. Thus:

$$u_i(B|P = 10) = 10 \times \mathbb{P}(B|P' = 100) = 10f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda}}$$

Consequently:

$$\mathbb{P}_i(B|P = 10) = \frac{e^{10\lambda_i f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda}}}}{e^{10\lambda_i f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda}}} + e^{\lambda_i}}$$

Given the distribution of type-i players, the average probability to buy when $P = 10$ is:

$$\begin{aligned} \mathbb{P}(B|P = 10) &= \sum_{\lambda_2=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \mathbb{P}_i(B|P = 10)\mathbb{P}(\lambda_i = \lambda_2) \\ &= f \sum_{\lambda_2=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{10\lambda_2 f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda}}}}{e^{10\lambda_2 f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda}}} + e^{\lambda_2}} \end{aligned}$$

When $\Lambda > 0$, some traders may buy the asset when $P = 100$. Thus buying generates a larger expected profit when $P = 10$, therefore more traders buy at this price: there is again a snowballing effect.

Consider finally the case of a trader observing a price $P = 1$. This trader perfectly infers from this observation that he is third in the sequence. The expected payoff of a player of type i if he buys, u_i, depends on the average probability to buy of players observing a price $P' = 10P = 10$. We have already defined this probability above, namely $P(B|P' = 10)$. Thus:

$$u_i(B|P = 1) = 10 \times \mathbb{P}(B|P' = 10) = 10f \sum_{\lambda_2=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{10\lambda_2 f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda}}}}{e^{10\lambda_2 f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda}}} + e^{\lambda_2}}$$

Given the distribution of type-i players, the average probability to buy when $P = 1$ is finally:

$$\begin{aligned}
\mathbb{P}(B|P=1) &= \sum_{\lambda_3=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \mathbb{P}_i(B|P=1)\mathbb{P}(\lambda_i=\lambda_3) \\
&= f \sum_{\lambda_3=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{\lambda_3 \times 10 f \sum_{\lambda_2=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{10\lambda_2 f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda}}}}{e^{10\lambda_2 f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda}}}+e^{\lambda_2}}}}{e^{\lambda_3 \times 10 f \sum_{\lambda_2=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{10\lambda_2 f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda}}}}{e^{10\lambda_2 f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda}}}+e^{\lambda_2}}} + e^{\lambda_3}}
\end{aligned}$$

4.3.2 No cap

Consider first the case of a trader observing a price $P \geq 100$. Let p^e be his expectation on the probability with which other traders, observing $P' \geq 100$, buy. His expected profit if he buys write:

$$u_i(B|P \geq 100) = 10 \times \frac{3}{7} \times p^e$$

His probability to buy is then:

$$\mathbb{P}_i(B|P \geq 100) = \frac{e^{\lambda_i \frac{30p^e}{7}}}{e^{\lambda_i \frac{30p^e}{7}} + e^{\lambda_i}}$$

Given the distribution of traders' types:

$$\begin{aligned}
\mathbb{P}(B|P \geq 100) &= \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \mathbb{P}_i(B|P \geq 100) f(\lambda) \\
&= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{\lambda_i \frac{30p^e}{7}}}{e^{\lambda_i \frac{30p^e}{7}} + e^{\lambda_i}}
\end{aligned}$$

Thus the overall probability to buy p is a fixed point:

$$p = f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{\lambda 10 \times \frac{3}{7} \times p}}{e^{\lambda 10 \times \frac{3}{7} \times p} + e^{\lambda}}.$$

Consider now the case of a trader observing a price $P < 100$. Conditional on observing this price, traders have a probability 1 not to be third. The probability to buy of a type i player can be found as in the case where there is a cap.

4.4 An extension of the Cognitive Hierarchy model with overconfidence (OCH)

In this section, we extend the CH model to the case where the parameter θ can possibly take negative values. As in the CH model, traders differ in their level of sophistication s. Following Camerer, Ho and Chong (2004), we assume that s is distributed according to a Poisson distribution F with mean τ. Each player s thinks that he understands the game differently than other players, and forms truncated beliefs about the fraction of h-level players according to $g_s(h) = \frac{f(h)}{\sum_{i=0}^{\max(s-\theta,0)} f(i)}$.

For reasons of parsimony and comparability to CH, we assume the truncation error parameter θ is common to all traders, whatever their level of sophistication. We say that an agent is overconfident if the level of sophistication he expects in the population of players is lower than what it actually is. θ is then an index of overconfidence. When it is $-\infty$, there is no overconfidence: each player adequately perceives the proportion of each level of sophistication. When it is $+\infty$, there is maximal overconfidence: each players believes that all other players are level 0. OCH is a two-parameter model with a Poisson parameter, τ, and an imagination parameter, θ. For each price P and each level s, we therefore compute the player's expected utility if he buys, conditional on P and s, in order to determine the theoretical probability to buy for each price as a function of the model's parameters τ and θ.

4.4.1 K=1

Consider the environment in which the cap on the initial price is equal to $K = 1$. We derive the conditional probabilities to buy for risk-neutral traders observing prices of $P = 1$, $P = 10$ and $P = 100$ respectively, in the OCH model.

Consider first the case of a trader observing a price $P = 100$. This trader perfectly infers from this observation that he is third in the sequence. Consequently, in this model, he only buys if he is a level-0 player. Given that there is a fraction $f(0) = \frac{\tau^0 \times \exp(-\tau)}{0!}$ of such traders in the population, and given that these traders buy randomly with probability $\Pr(B|P = 100, s = 0) = \frac{1}{2}$, the probability to buy conditional on the price being $P = 100$ writes:

$$\mathbb{P}(B|P = 100) = \frac{1}{2}\exp(-\tau)$$

Consider now the case of a trader observing a price $P = 10$. This trader perfectly infers from this observation that he is second in the sequence.

- If he is a level-0 player, he buys with probability $\mathbb{P}(B|P = 10, s = 0) = \frac{1}{2}$.

- If he is a level-1 player, he thinks that the next player observing the price $P_3 = 10 \times P_2$ is a mixture of level-0, level-1, ..., level $1-\theta$. Consequently, his expected profit if he buys writes:

$$u_1(B|P = 10) = \left(\frac{1}{2} \times 10\right) \times \frac{f(0)}{\sum_{i=0}^{\max(1-\theta,0)} f(i)},$$

where $f(i) = e^{-\tau}\frac{\tau^i}{i!}$, while his profit if he does not buy is $u_\varnothing = 1$.

$$\begin{aligned} u_1(B|P &= 10, \theta \geq 1) = \left(\frac{1}{2} \times 10\right) > u_\varnothing \\ u_1(B|P &= 10, \theta < 1) = \left(\frac{1}{2} \times 10\right) \times \frac{f(0)}{f(0) + f(1) + ... + f(1-\theta)} \\ &= \frac{5}{1 + \tau + ... + \frac{\tau^{1-\theta}}{(1-\theta)!}} \end{aligned}$$

This shows that the parameter θ is not always identifiable. Indeed, when $\theta \geq 1$, the probability to buy is one for any value of θ.

When $\theta < 1$, buying is beneficial if:

$$u_1(B|P = 10, \theta < 1) > u_1(\oslash|P = 10, \theta < 1) \iff \frac{5}{1 + \tau + ... + \frac{\tau^{1-\theta}}{(1-\theta)!}} > 1$$

When $\theta > 1$ and $\tau < \ln 5$, the last inequality is satisfied for any value of θ so that this parameter is not identifiable. The same logic applies for all potential prices, step levels, and caps on prices. The threshold on the value of τ for which we cannot identify θ is different depending on the probability to resell.

The probability to buy is therefore:

$$\begin{aligned}\mathbb{P}(B|P &= 10, s=1) = 1 \text{ if } \theta \geq 1 \\ &= 1 \text{ if } \theta < 1 \text{ and } 1+\tau+...+\frac{\tau^{1-\theta}}{(1-\theta)!} < 5 \\ &= 0 \text{ if } \theta < 1 \text{ and } 1+\tau+...+\frac{\tau^{1-\theta}}{(1-\theta)!} \geq 5\end{aligned}$$

- More generally, if he is a level-s player with $s \geq 2$, he thinks that the next player observing the price $P_3 = 10 \times P_2$ is a mixture of level-0, level-1 ... level-$s-\theta$ players, precisely a level-0 with probability $f(0) = \exp(-\tau)$, a level-1 with probability $f(1) = \tau \times \exp(-\tau)$, ... and a level-$s-\theta$ player with the truncated probability $1 - \sum_{i=0}^{s-\theta} f(i)$. Given that he expects level-s-1 players (for $s \geq 2$) not to buy at price 100, his expected profit if he buys writes:

$$u_{s\geq 2}(B|P=10) = \left(\frac{f(0)}{\sum_{i=0}^{\max(s-\theta,0)} f(i)} \times \frac{1}{2}\right) \times 10$$

where $f(i) = e^{-\tau}\frac{\tau^i}{i!}$.

$$u_{s\geq 2}(B|P=10) = \frac{1}{\sum_{i=0}^{\max(s-\theta,0)} \frac{\tau^i}{i!}} \times \frac{1}{2} \times 10,$$

The probability with which the trader $s \geq 2$ buys conditional on observing $P = 10$ writes:

$$\begin{aligned}\mathbb{P}(B|P &= 10, s) = 1 \text{ if } \sum_{i=0}^{\max(s-\theta,0)} \frac{\tau^i}{i!} < 5 \\ &= 0 \text{ otherwise}\end{aligned}$$

Finally, given the distribution of players, and since $f(0) = e^{-\tau}$, the probability to buy conditional on the price being $P = 10$ writes:

$$\mathbb{P}(B|P=10) = e^{-\tau}\left(\frac{1}{2} + \sum_{s=1}^{\infty} \frac{\tau^s}{s!} \times 1_{\sum_{i=0}^{\max(s-\theta,0)} \frac{\tau^i}{i!} < 5}\right).$$

Since we cannot numerically compute this infinite sum, we stop numerically at $s = 100$.

Consider finally the case of a trader observing a price $P = 1$. This trader perfectly infers from this observation that he is first in the sequence.

- If he is a level-0 player, he buys with probability $\mathbb{P}\left(B|P = 1, s = 0\right) = \frac{1}{2}$.

- If he is a level-s player with $s \geq 1$, he thinks that the next player observing the price $P_2 = 10 \times P_1$ is a mixture of level-0, level-1 ... level-$s - \theta$ players. His expected profit writes:

$$u_{s\geq 1}(B|P = 1) = \frac{\sum_{j=0}^{\max(s-\theta,0)} f(j)\mathbb{P}(B|P = 10, \tilde{S} = j)}{\sum_{i=0}^{\max(s-\theta,0)} f(i)} \times 10$$

which simplifies to:

$$u_{s\geq 1}(B|P = 1) = \frac{\sum_{j=0}^{\max(s-\theta,0)} \frac{\tau^j}{j!}\mathbb{P}(B|P = 10, s = j)}{\sum_{i=0}^{\max(s-\theta,0)} \frac{\tau^i}{i!}} \times 10.$$

If $s - \theta > 0$, then:

$$u_{s\geq 1, s-\theta>0}(B|P = 1) = \frac{\frac{1}{2} + \sum_{j=1}^{s-\theta} \frac{\tau^j}{j!} \times 1_{\sum_{i=0}^{\max(j-\theta,0)} \frac{\tau^i}{i!} < 5}}{\sum_{i=0}^{\max(s-\theta,0)} \frac{\tau^i}{i!}} \times 10.$$

else if $s - \theta \leq 0$ then:

$$u_{s\geq 1, s-\theta\leq 0}(B|P = 1) = \frac{1}{2} \times 10.$$

Again, this expected profit depends on the value of τ and θ. The probability to buy for $s \geq 1$ is therefore:

$$\begin{aligned} \mathbb{P}(B|P &= 1, s) = 1 \text{ if } s - \theta \leq 0. \\ &= 1 \text{ if } s - \theta > 0 \text{ and } 4 > \sum_{j=1}^{s-\theta} \frac{\tau^j}{j!} \left(1 - 10 \times 1_{\sum_{i=0}^{\max(j-\theta,0)} \frac{\tau^i}{i!} < 5}\right) \end{aligned}$$

Finally, given the distribution of players, the probability to buy conditional on the price being $P = 1$ writes:

$$\mathbb{P}(B|P = 1) = e^{-\tau} \left(\frac{1}{2} + \sum_{s=1}^{\infty} \left(\frac{\tau^s}{s!} \times \mathbb{P}(B|P = 1, s) \right) \right).$$

Probabilities are a function of τ and θ.

4.4.2 No cap and $\theta > 0$

Consider now an environment in which there is no cap on the initial price. Suppose first that $\theta > 0$, so that players believe that other players are less sophisticated than they are.
Consider first the case of a trader observing a price $P \geq 100$.

- If he is a level-0 player, he buys with probability $\mathbb{P}(B|P \geq 100, s = 0) = \frac{1}{2}$.

- If he is a level-s player, with $0 < s \leq \theta$, he thinks that the next player observing the price $P' = 10P$ is a level-0 player with probability 1. Given that he is not last with probability $\frac{3}{7}$, his expected profit writes:

$$u_{0<s\leq\theta}(B|P \geq 100) = 10 \times \frac{3}{7} \times \frac{1}{2},$$

which is strictly higher than his expected utility if he does not buy. His probability to buy is therefore equal to 1.

- If he is a level-s player, with $s > \theta$, he thinks that the next player observing the price $P' = 10P$ is a mixture of level-0, level-1 ... level-$s - \theta$ players. Given that he is not last with probability $\frac{3}{7}$, his expected profit writes:

$$u_{s>\theta}(B|P \geq 100) = 10 \times \frac{3}{7} \times \frac{\sum_{i=0}^{s-\theta} \mathbb{P}(B|P' \geq 100, s = i) \times f(i)}{\sum_{j=0}^{s-\theta} f(j)}$$

It can recursively be shown (starting with $s = \theta + 1$) that since level-i players for $0 < i \leq \theta$ buy when they observe $P' = 10P$, it is strictly profitable for the level-s player to buy when he observes $P' = P$, for $s > \theta$.

Consequently, whatever θ, the probability to buy conditional on $P \geq 100$ writes:

$$\mathbb{P}(B|P \geq 100) = 1 - e^{-\tau}\frac{1}{2}$$

Since it is profitable for level-1 players to buy, it becomes profitable for more sophisticated players to buy, thus only level-0 players do not buy.
Consider now the case of a trader observing $P = 10$ or $P = 1$. Since only level-0 players do not buy when they observe $P = 100$ ($P = 10$ respectively), it becomes even more profitable for all non level-0 players to buy when they observe $P = 10$ ($P = 1$ respectively).

We thus conclude that the parameter θ is indeterminate. When there is no cap, the OCH model with $\theta > 0$ has the same predictions as the CH model: the probability to buy is constant whatever the observed price, and equal to $\mathbb{P}(B|P) = 1 - e^{-\tau}\frac{1}{2}$.

4.4.3 No cap and $\theta \leq 0$

Consider again an environment in which there is no cap on the initial price, but suppose now that $\theta \leq 0$. In this case, players believe that other players may be more sophisticated than they are. This case is slightly more complex, as players have to form beliefs on the behavior of these more sophisticated players. We restrict our analysis to the two following monotonic beliefs' specifications:

1. Level-s player expects all level-s players with $s' \geq 1$ to buy when they observe $P' \geq 100$.
2. Level-s player thinks that there exists a threshold $s^*_\oslash$ such that all level-s' players observing $P' \geq 100$ buy if $s' \leq s^*_\oslash$ and do not buy if $s' > s^*_\oslash$.

Under the first specification of beliefs

Consider first the case of a trader observing $P \geq 100$. Let us define:

$$p_s = \mathbb{P}(B|P' \geq 100, s).$$

- If he is a level-0 player, he buys with probability $\mathbb{P}(B|P \geq 100, s = 0) = \frac{1}{2}$.

- If he is a level-s player, with $s \geq 1$, he thinks that the next player observing the price $P' = 10P$ is a mixture of level-0, level-1 ... level-$s - \theta$ players. Given that he is not last with probability $\frac{3}{7}$, his expected profit writes:

$$u_{s\geq 1}(B|P \geq 100) = 10 \times \frac{3}{7} \times \frac{\sum_{i=0}^{s-\theta} p_i \times f(i)}{\sum_{j=0}^{s-\theta} f(j)}$$

This yields:

$$\begin{aligned} u_s(B|P \geq 100) &= 10 \times \frac{3}{7} \times \frac{\sum_{i=0}^{s-\theta} p_i \times f(i)}{\sum_{j=0}^{s-\theta} f(j)} \\ &= u_{s+1}(B|P' \geq 100) + \frac{(u_{s+1}(B|P' \geq 100) - \frac{30}{7}p_{s+1-\theta})f(s+1-\theta)}{\sum_{j=0}^{s-\theta} f(j)} \end{aligned}$$

and conversely:

$$\begin{aligned}u_{s+1}(B|P \geq 100) &= 10 \times \frac{3}{7} \times \frac{\sum_{i=0}^{s+1-\theta} p_i \times f(i)}{\sum_{j=0}^{s+1-\theta} f(j)} \\ &= u_s(B|P' \geq 100) - \frac{(u_s(B|P' \geq 100) - \frac{30}{7} p_{s+1-\theta}) f(s+1-\theta)}{\sum_{j=0}^{s+1-\theta} f(j)}\end{aligned}$$

Consider the first beliefs' specification. Let us show that beliefs on actions are consistent with actual choices, namely, that for $s \geq 1$, if a level-s player expects all level-s' players (for $s' \geq 1$) to buy when they observe $P' \geq 100$, then it is profitable for him to buy.

Under these beliefs on actions, his expected profit if he buys writes:

$$u_{s \geq 1}(B|P \geq 100) = 10 \times \frac{3}{7} \times \frac{f(0)\frac{1}{2} + \sum_{i=1}^{s-\theta} f(i)}{\sum_{j=0}^{s-\theta} f(j)}$$

We consequently have:

$$u_{s \geq 1}(B|P \geq 100) > u_{\oslash} \iff \frac{8}{7} + \sum_{i=1}^{s-\theta} \frac{\tau^j}{j!} > 0,$$

which always holds, whatever τ and θ. Under this specification, θ is thus indeterminate and only level-0 player do not buy with probability $\frac{1}{2}$, so that:

$$\mathbb{P}(B|P \geq 100) = 1 - \frac{1}{2} e^{-\tau}.$$

Consider now the case of a trader observing $P = 10$ or $P = 1$. Under the first beliefs' specification, the probability not to be last of this trader is equal to 1. Consequently, if all level-s players with $s \geq 1$ buy when they observe $P = 100$, it is even more profitable to buy for the level-s player with $s \geq 1$ when he observes $P = 10$. The same reasoning holds for the level-s player with $s \geq 1$ when he observes $P = 1$. This yields:

$$\mathbb{P}(B|P = 10) = \mathbb{P}(B|P = 1) = 1 - \frac{1}{2} e^{-\tau}.$$

Under the second specification of beliefs
Consider first the case of a trader observing $P \geq 100$. Consider the second

beliefs' specification. Let us show that beliefs on actions are consistent with actual choices, and let us find $s^*_\oslash$. If level-s player thinks that there exists a threshold $s^*_\oslash$ such that all level-s' players observing $P' \geq 100$ buy if $s' \leq s^*_\oslash$ and do not buy if $s' > s^*_\oslash$, his expected profit if he buys writes:

$$u_{s\geq 1}(B|P \geq 100) = 10 \times \frac{3}{7} \times \frac{\frac{1}{2}f(0) + \sum_{i=1}^{\min(s-\theta, s^*_\oslash)} f(i)}{\sum_{j=0}^{s-\theta} f(j)}$$

Let us first show that if $p_{s^*_\oslash+1} = 0$, then $p_s = 0$ for all $s \geq s^*_\oslash + 1$. In this case, $\min(s-\theta, s^*_\oslash) = s^*_\oslash$ and :

$$u_{s+1}(B|P \geq 100) = u_s(B|P' \geq 100) - \frac{u_s(B|P' \geq 100)f(s+1-\theta)}{\sum_{j=0}^{s+1-\theta} f(j)},$$

so that the expected utility of level-(s+1) player if he buys is even lower than the expected utility of buying of a level-s player. Consequently, if level-s player does not buy, he does not buy either.

Let us now show that if $p_{s^*_\oslash} = 1$, then $p_s = 1$ for all $s \leq s^*_\oslash$. If $s \leq s^*_\oslash + \theta$, then we have shown that $u_{s\geq 1}(B|P \geq 100) > u_\oslash$. Conversely, if $s > s^*_\oslash + \theta$, then $\min(s-\theta, s^*_\oslash) = s^*_\oslash$ and:

$$u_{s-1}(B|P \geq 100) = u_s(B|P' \geq 100) + \frac{u_s(B|P' \geq 100)f(s-\theta)}{\sum_{j=0}^{s-1-\theta} f(j)},$$

Consequently, if level-s player buys, level-(s-1) player buys as well.

Let us finally show how to find the threshold $s^*_\oslash$. If it exists and is finite, it must be such that:

$$u_{s^*_\oslash}(B|P \geq 100) = \frac{30}{7} \times \frac{\frac{1}{2} + \sum_{i=1}^{s^*_\oslash} \frac{\tau^i}{i!}}{\sum_{j=0}^{s^*_\oslash-\theta} \frac{\tau^j}{j!}} > 1$$

and

$$u_{s^*_\oslash+1}(B|P \geq 100) = \frac{30}{7} \times \frac{\frac{1}{2} + \sum_{i=1}^{s^*_\oslash} \frac{\tau^i}{i!}}{\sum_{j=0}^{s^*_\oslash+1-\theta} \frac{\tau^j}{j!}} < 1$$

Notice that if τ is sufficiently low, the second condition will never be satisfied, so that $s^*_\oslash \to \infty$. The latter case collapses to our first beliefs' specification, where θ is indeterminate.

Finally, under this beliefs' specification, if there exists a finite $s^*_\oslash$, then:

$$P(B|P \geq 100) = \frac{1}{2}e^{-\tau} + \sum_{j=1}^{s^*_\oslash} \frac{\tau^j}{j!}e^{-\tau}$$

Consider now the case of a trader observing $P = 10$. Level-s player's expected profit if he buys writes:

$$u_{s\geq 1}(B|P \geq 100) = 10 \times \frac{\frac{1}{2}f(0) + \sum_{i=1}^{\min(s-\theta, s^*_\oslash)} f(i)}{\sum_{j=0}^{s-\theta} f(j)}$$

Consequently, if level-s players buy when they observe $P' = 100$, it is even more profitable for the level-s player to buy when he observes $P = 10$. Still, there can be a higher threshold $s^{**}_\oslash > s^*_\oslash$ such that level-s players observing $P = 10$ do not buy if $s > s^{**}_\oslash$.

If this threshold exists and is finite, it must be such that:

$$u_{s^{**}_\oslash}(B|P = 10) > 1$$

and

$$u_{s^{**}_\oslash + 1}(B|P = 10) < 1$$

Finally, under this beliefs' specification, if there exists a finite $s^{**}_\oslash$, then:

$$P(B|P = 10) = \frac{1}{2}e^{-\tau} + \sum_{j=1}^{s^{**}_\oslash} \frac{\tau^j}{j!}e^{-\tau}$$

Consider now the case of a trader observing $P = 1$. Level-s player's expected profit if he buys writes:

$$u_{s\geq 1}(B|P \geq 100) = 10 \times \frac{\frac{1}{2}f(0) + \sum_{i=1}^{\min(s-\theta, s^{**}_\oslash)} f(i)}{\sum_{j=0}^{s-\theta} f(j)}$$

Consequently, if level-s players buy when they observe $P' = 10$, it is even more profitable for the level-s player to buy when he observes $P = 1$. Still, there can be a higher threshold $s^{***}_\oslash > s^{**}_\oslash$ such that level-s players observing $P = 1$ do not buy if $s > s^{***}_\oslash$.

If this threshold exists and is finite, it must be such that:

$$u_{s_{\oslash}^{***}}(B|P=1) > 1$$

and

$$u_{s_{\oslash}^{***}+1}(B|P=1) < 1$$

Finally, under this beliefs' specification, if there exists a finite $s_{\oslash}^{***}$, then:

$$P(B|P=1) = \frac{1}{2}e^{-\tau} + \sum_{j=1}^{s_{\oslash}^{***}} \frac{\tau^j}{j!}e^{-\tau}$$

5 The Analogy-Based Expectation Equilibrium of Jehiel, 2005

According to the ABEE logic, agents use simplified representations of their environment in order to form expectations. In particular, agents are assumed to bundle nodes at which other agents make choices into analogy classes. Agents then form correct beliefs concerning the average behavior within each analogy class. Following Huck, Jehiel, and Rutter (2010), we consider that agents apply noisy best-responses to their beliefs.

In our bubble game, two types of analogy classes arise naturally. On the one hand, traders may use only one analogy class, assuming that other traders' behavior is the same across all potential prices. On the other hand, traders may use two analogy classes: one to form beliefs regarding the behavior of traders who are sure not to be last in the market sequence (Class I), the other to form beliefs regarding the behavior of the remaining traders (Class II that thus includes traders who think they may be last or who know they are last).

In this section, we derive the conditional probabilities to buy for risk-neutral traders observing prices of $P \in \{1, 10...\}$ in the ABEE model of Jehiel (2005). Let $u_{i,B}$ be the expected payoff of risk-neutral player observing $P = P_i$ if he buys, $u_{\varnothing}$ his expected payoff if he does not buy. In the quantal response model, the probability with which the trader buys conditional on observing P writes:

$$\mathbb{P}(B|P=P_i) = \frac{e^{\Lambda u_{i,B}}}{e^{\Lambda u_{i,B}} + e^{\Lambda u_{i,\varnothing}}}$$

5.1 K=1

Consider first an environment in which there is a cap $K = 1$ on the initial price. There are three possible prices. Given the probability distribution of the first price, we have:

$$\begin{aligned} P(\text{I observe } 1) &= \frac{1}{3} \\ P(\text{I observe } 10) &= \frac{1}{3} \\ P(\text{I observe } 100) &= \frac{1}{3} \end{aligned}$$

Let p_1, p_2, and p_3 denote the actual probability that a trader buys after observing prices equal to 1, 10, and 100, respectively. Let $\mathbb{P}(B|P = 1)$, $\mathbb{P}(B|P = 10)$, and $\mathbb{P}(B|P = 100)$ be the corresponding probabilities as (mis)perceived by traders using analogy classes. In the one-class ABEE, players bundle probabilities to buy so that:

$$\mathbb{P}(B|P = 1) = \mathbb{P}(B|P = 10) = \mathbb{P}(B|P = 100) = \frac{\frac{1}{3}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_3}{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = \frac{p_1 + p_2 + p_3}{3},$$

while in the two-class ABEE,

$$\begin{aligned} \text{Class II} &: \quad \mathbb{P}(B|P = 100) = p_3 \\ \text{Class I} &: \quad \mathbb{P}(B|P = 1) = \mathbb{P}(B|P = 10) = \frac{\frac{1}{3}p_1 + \frac{1}{3}p_2}{\frac{1}{3} + \frac{1}{3}} = \frac{p_1 + p_2}{2}. \end{aligned}$$

5.1.1 K=1, 1 class ABEE

Consider the case where there exists only one analogy class.
Consider first the case of a trader observing a price $P = 100$. This trader perfectly infers from this observation that he is third in the sequence, that is, $q(1, 100) = 0$. Consequently, his expected payoffs if he buys writes:

$$u_{3,B} = 0,$$

so that his probability to buy is:

$$p_3 = \frac{1}{1 + e^{\Lambda}}$$

Consider now the case of a trader observing a price $P = 10$. Given that he knows that he is second, that is, $q(1,10) = 1$, his expected payoffs if he buys writes:

$$u_{2,B} = 10 \times \mathbb{P}(B|P = 100) = 10(\frac{p_1 + p_2 + p_3}{3})$$

The probability to buy is therefore:

$$p_2 = \frac{e^{\Lambda \frac{10(p_1+p_2+p_3)}{3}}}{e^{\Lambda \frac{10(p_1+p_2+p_3)}{3}} + e^{\Lambda}}$$

Consider finally the case of a trader observing a price $P = 1$. Given that he knows that he is first, that is, $q(1,1) = 1$, his expected payoffs if he buys writes:

$$u_{1,B} = 10 \times \mathbb{P}(B|P = 10) = \frac{10}{3}(p_1 + p_2 + p_3)$$

The probability to buy is therefore:

$$p_1 = p_2$$

Consequently, in equilibrium:

$$\begin{aligned}
p_3 &= \frac{1}{1 + e^{\Lambda}} \\
p_2 &= \frac{e^{\Lambda \frac{10(\frac{1}{1+e^{\Lambda}} + 2p_2)}{3}}}{e^{\Lambda \frac{10(\frac{1}{1+e^{\Lambda}} + 2p_2)}{3}} + e^{\Lambda}} \\
p_1 &= p_2
\end{aligned}$$

Solving this system enables us to find, for each j, p_j as a function of λ.

5.1.2 K=1, 2 classes (1 10)(100)

Consider now the case where there exists two analogy classes I and II.
Consider first the case of a trader observing a price $P = 100$. Given that $q(1, 100) = 0$, his expected payoffs for buying writes:

$$u_{3,B} = 0$$

The probability to buy is therefore:

$$p_3 = \frac{1}{1 + e^{\Lambda}}$$

Consider now the case of a trader observing a price $P = 10$. Given that $q(1,10) = 1$, his expected payoffs for buying write:

$$u_{2,B} = 10 \times \mathbb{P}(B|P = 100) = 10p_3$$

The probability to buy is therefore:

$$p_2 = \frac{e^{\Lambda 10 p_3}}{e^{\Lambda 10 p_3} + e^{\Lambda}}$$

Consider finally the case of a trader observing a price $P = 1$. Given that $q(1,1) = 1$, his expected payoffs for buying writes:

$$u_{1,B} = 10 \times \mathbb{P}(B|P = 10) = 10(\frac{p_1 + p_2}{2})$$

The probability to buy is therefore:

$$p_1 = \frac{e^{\Lambda 5(p_1+p_2)}}{e^{\Lambda 5(p_1+p_2)} + e^{\Lambda}}$$

Consequently, in equilibrium:

$$\begin{aligned}
p_3 &= \frac{1}{1+e^{\Lambda}} \\
p_2 &= \frac{e^{\Lambda 10 p_3}}{e^{\Lambda 10 p_3} + e^{\Lambda}} \\
p_1 &= \frac{e^{5\Lambda(p_1+p_2)}}{e^{5\Lambda(p_1+p_2)} + e^{\Lambda}}
\end{aligned}$$

5.2 K=100

Consider now an environment in which there is a cap $K = 100$ on the initial price. There are five possible prices. Given the probability distribution of the first price, we have:

$$\begin{aligned}
\mathbb{P}(\text{I observe } 1) &= \frac{1}{6} \\
\mathbb{P}(\text{I observe } 10) &= \frac{1}{4} \\
\mathbb{P}(\text{I observe } 100) &= \frac{1}{3} \\
\mathbb{P}(\text{I observe } 1000) &= \frac{1}{6} \\
\mathbb{P}(\text{I observe } 10000) &= \frac{1}{12}
\end{aligned}$$

Besides, recall that:

$$\begin{aligned} q(100,1) &= q(100,10) = 1 \\ q(100,100) &= q(100,1000) = \frac{1}{2} \\ q(100,10000) &= 0 \end{aligned}$$

Let p_1, p_2, p_3, p_4 and p_5 denote the actual probability that a trader buys after observing prices equal to 1, 10, 100, 1,000 and 10,000 respectively. Let $\mathbb{P}(B|P=1)$, $\mathbb{P}(B|P=10)$, $\Pr(B|P=100)$, $\mathbb{P}(B|P=1000)$, and $\mathbb{P}(B|P=10000)$ be the corresponding probabilities as (mis)perceived by traders using analogy classes. In the one-class ABEE, players bundle probabilities to buy so that:

$$\mathbb{P}(B|P) = \frac{\frac{1}{6}p_1 + \frac{1}{4}p_2 + \frac{1}{3}p_3 + \frac{1}{6}p_4 + \frac{1}{12}p_5}{\frac{1}{6} + \frac{1}{4} + \frac{1}{3} + \frac{1}{6} + \frac{1}{12}} = \frac{2p_1 + 3p_2 + 4p_3 + 2p_4 + p_5}{12}, \forall P$$

while in the two-class ABEE,

$$\begin{aligned} \text{Class II} \quad &: \quad \mathbb{P}(B|P=100,1000,10000) = \frac{\frac{1}{3}p_3 + \frac{1}{6}p_4 + \frac{1}{12}p_5}{\frac{1}{3} + \frac{1}{6} + \frac{1}{12}} = \frac{4p_3 + 2p_4 + p_5}{7} \\ \text{Class I} \quad &: \quad \mathbb{P}(B|P=1,10) = \frac{\frac{1}{6}p_1 + \frac{1}{4}p_2}{\frac{1}{6} + \frac{1}{4}} = \frac{2p_1 + 3p_2}{5}. \end{aligned}$$

5.2.1 K=100, 1 class

Consider the case where there exists only one analogy class.
Consider first the case of a trader observing a price $P = 10,000$. Given that $q(100,10000) = 0$, his expected payoffs for buying writes:

$$u_{5,B} = 0$$

The probability to buy is therefore:

$$p_5 = \frac{1}{1+e^{\Lambda}}$$

Consider now the case of a trader observing a price $P = 1,000$. Given that $q(100,1000) = \frac{1}{2}$, his expected payoffs for buying writes:

$$u_{4,B} = \frac{1}{2} \times 10 \times \mathbb{P}(B|P=10,000) = 5\frac{2p_1 + 3p_2 + 4p_3 + 2p_4 + p_5}{12}$$

The probability to buy is therefore:

$$p_4 = \mathbb{P}\left(B|P=1,000\right) = \frac{e^{\Lambda\frac{5(2p_1+3p_2+4p_3+2p_4+p_5)}{12}}}{e^{\Lambda\frac{5(2p_1+3p_2+4p_3+2p_4+p_5)}{12}}+e^{\Lambda}}$$

Consider now the case of a trader observing a price $P=100$. Given that $q(100,100)=\frac{1}{2}$, his expected payoffs for buying writes:

$$u_{3,B} = \frac{1}{2}\times 10\times\mathbb{P}(B|P=1,000) = 5\frac{2p_1+3p_2+4p_3+2p_4+p_5}{12}$$

The probability to buy is therefore:

$$p_3 = p_4$$

Consider now the case of a trader observing a price $P=10$. Given that $q(100,10)=1$, his expected payoffs for buying writes:

$$u_{2,B} = 1\times 10\times\mathbb{P}(B|P=100) = 10\frac{2p_1+3p_2+4p_3+2p_4+p_5}{12}$$

The probability to buy is therefore:

$$p_2 = \frac{e^{\Lambda\frac{5(2p_1+3p_2+4p_3+2p_4+p_5)}{6}}}{e^{\Lambda\frac{5(2p_1+3p_2+4p_3+2p_4+p_5)}{6}}+e^{\Lambda}}$$

Consider finally the case of a trader observing a price $P=1$. Given that $q(100,1)=1$, his expected payoffs for buying writes:

$$u_{1,B} = 10\times\mathbb{P}(B|P=10) = 10\frac{2p_1+3p_2+4p_3+2p_4+p_5}{12}$$

The probability to buy is therefore:

$$p_1 = p_2$$

Consequently, in equilibrium:

$$\begin{aligned}
p_5 &= \frac{1}{1+e^{\Lambda}}\\
p_4 &= \frac{e^{\Lambda\frac{5(5p_2+6p_4+\frac{1}{1+e^{\Lambda}})}{12}}}{e^{\Lambda\frac{5(5p_2+6p_4+\frac{1}{1+e^{\Lambda}})}{12}}+e^{\Lambda}}\\
p_3 &= p_4\\
p_2 &= \frac{e^{\Lambda\frac{5(5p_2+6p_4+\frac{1}{1+e^{\Lambda}})}{6}}}{e^{\Lambda\frac{5(5p_2+6p_4+\frac{1}{1+e^{\Lambda}})}{6}}+e^{\Lambda}}\\
p_1 &= p_2
\end{aligned}$$

5.2.2 K=100, 2 classes (1 10) (100 1,000 10,000)

Consider now the case where there exists two analogy classes, namely (1 10) and (100 1,000 10,000).
Consider first the case of a trader observing a price $P = 10,000$. Given that $q(100, 10000) = 0$, his expected payoffs for buying writes:

$$u_{5,B} = 0$$

His probability to buy is therefore:

$$p_5 = \frac{1}{1 + e^{\Lambda}}$$

Consider now the case of a trader observing a price $P = 1,000$. Given that $q(100, 1000) = \frac{1}{2}$, his expected payoffs for buying writes:

$$u_{4,B} = \frac{1}{2} \times 10 \times \mathbb{P}(B|P = 10,000) = 5\frac{4p_3 + 2p_4 + p_5}{7}$$

The probability to buy is therefore:

$$p_4 = \mathbb{P}\left(B|P = 1,000\right) = \frac{e^{\Lambda\frac{5(4p_3+2p_4+p_5)}{7}}}{e^{\Lambda\frac{5(4p_3+2p_4+p_5)}{7}} + e^{\Lambda}}$$

Consider now the case of a trader observing a price $P = 100$. Given that $q(100, 100) = \frac{1}{2}$, his expected payoffs for buying writes:

$$u_{3,B} = \frac{1}{2} \times 10 \times \mathbb{P}(B|P = 1,000) = 5\frac{4p_3 + 2p_4 + p_5}{7}$$

The probability to buy is therefore:

$$p_3 = p_4$$

Consider now the case of a trader observing a price $P = 10$. Given that $q(100, 10) = 1$, his expected payoffs for buying writes:

$$u_{2,B} = 10 \times \mathbb{P}(B|P = 100) = 10\frac{4p_3 + 2p_4 + p_5}{7}$$

The probability to buy is therefore:

$$p_2 = \frac{e^{\Lambda\frac{10(4p_3+2p_4+p_5)}{7}}}{e^{\Lambda\frac{10(4p_3+2p_4+p_5)}{7}} + e^{\Lambda}}$$

Consider finally the case of a trader observing a price $P = 1$. His expected payoffs for buying and not buying respectively write:

$$u_{1,B} = 10 \times \mathbb{P}(B|P = 10) = 10\frac{2p_1 + 3p_2}{5}$$

The probability to buy is therefore:

$$p_1 = \mathbb{P}\left(B|P = 1\right) = \frac{e^{\Lambda(4p_1+6p_2)}}{e^{\Lambda(4p_1+6p_2)} + e^{\Lambda}}$$

Consequently, in equilibrium:

$$\begin{aligned}
p_5 &= \frac{1}{1+e^{\Lambda}} \\
p_4 &= \frac{e^{\Lambda\frac{5(6p_4+\frac{1}{1+e^{\Lambda}})}{7}}}{e^{\Lambda\frac{5(6p_4+\frac{1}{1+e^{\Lambda}})}{7}} + e^{\Lambda}} \\
p_3 &= p_4 \\
p_2 &= \frac{e^{\Lambda\frac{10(6p_4+\frac{1}{1+e^{\Lambda}})}{7}}}{e^{\Lambda\frac{10(6p_4+\frac{1}{1+e^{\Lambda}})}{7}} + e^{\Lambda}} \\
p_1 &= \frac{e^{\Lambda(4p_1+6p_2)}}{e^{\Lambda(4p_1+6p_2)} + e^{\Lambda}}
\end{aligned}$$

5.3 K=10,000

Consider now an environment in which there is a cap $K = 10000$ on the initial price. There are seven possible prices. Given the probability distribution of the first price, we have:

$$
\begin{aligned}
\mathbb{P}(\text{I observe } 1) &= \frac{1}{6} \\
\mathbb{P}(\text{I observe } 10) &= \frac{1}{4} \\
\mathbb{P}(\text{I observe } 100) &= \frac{7}{24} \\
\mathbb{P}(\text{I observe } 1000) &= \frac{7}{48} \\
\mathbb{P}(\text{I observe } 10,000) &= \frac{1}{12} \\
\mathbb{P}(\text{I observe } 100,000) &= \frac{1}{24} \\
\mathbb{P}(\text{I observe } 1,000,000) &= \frac{1}{48}
\end{aligned}
$$

Besides, recall that:

$$
\begin{aligned}
q(10000,1) &= q(10000,10) = 1 \\
q(10000,100) &= q(10000,1000) = \frac{3}{7} \\
q(10000,10000) &= q(10000,100000) = \frac{1}{2} \\
q(10000,1000000) &= 0
\end{aligned}
$$

Let p_1, p_2, p_3, p_4, p_5, p_6 and p_7 denote the actual probability that a trader buys after observing prices equal to 1, 10, 100, 1,000, 10,000, 100,000 and 1,000,000 respectively. Let $\mathbb{P}(B|P=1)$, $\mathbb{P}(B|P=10)$, $\mathbb{P}(B|P=100)$, $\mathbb{P}(B|P=1000)$, $\mathbb{P}(B|P=10000)$, $\Pr(B|P=100000)$ and $\mathbb{P}(B|P=1000000)$ be the corresponding probabilities as (mis)perceived by traders using analogy classes. In the one-class ABEE, players bundle probabilities to buy so that:

$$
\begin{aligned}
\mathbb{P}(B|P) &= \frac{\frac{1}{6}p_1 + \frac{1}{4}p_2 + \frac{7}{24}p_3 + \frac{7}{48}p_4 + \frac{1}{12}p_5 + \frac{1}{24}p_6 + \frac{1}{48}p_7}{\frac{1}{6} + \frac{1}{4} + \frac{7}{24} + \frac{7}{48} + \frac{1}{12} + \frac{1}{24} + \frac{1}{48}} \\
&= \frac{8p_1 + 12p_2 + 14p_3 + 7p_4 + 4p_5 + 2p_6 + p_7}{48}, \forall P
\end{aligned}
$$

while in the two-class ABEE,

Class II :
$$\mathbb{P}(B|P \geq 100) = \frac{\frac{7}{24}p_3 + \frac{7}{48}p_4 + \frac{1}{12}p_5 + \frac{1}{24}p_6 + \frac{1}{48}p_7}{\frac{7}{24} + \frac{7}{48} + \frac{1}{12} + \frac{1}{24} + \frac{1}{48}} = \frac{14p_3 + 7p_4 + 4p_5 + 2p_6 + p_7}{28}$$
$$\text{Class I}: \ \mathbb{P}(B|P = 1, 10) = \frac{\frac{1}{6}p_1 + \frac{1}{4}p_2}{\frac{1}{6} + \frac{1}{4}} = \frac{2p_1 + 3p_2}{5}.$$

5.3.1 K=10,000, 1 class

Consider the case where there exists only one analogy class.
Consider first the case of a trader observing a price $P = 1,000,000$. Given that $q(10000, 1000000) = 0$, his expected payoffs for buying writes:

$$\begin{aligned} u_{7,B} &= 0 \\ u_{\varnothing} &= 1 \end{aligned}$$

The probability to buy is therefore:

$$p_7 = \frac{1}{1 + e^{\Lambda}}$$

Consider now the case of a trader observing a price $P = 100,000$. His expected payoffs for buying writes:

$$u_{6,B} = \frac{1}{2} \times 10 \times \mathbb{P}(B|P = 1,000,000) = 5(\frac{8p_1 + 12p_2 + 14p_3 + 7p_4 + 4p_5 + 2p_6 + p_7}{48})$$

The probability to buy is therefore:

$$p_6 = \frac{e^{\Lambda\frac{5(8p_1+12p_2+14p_3+7p_4+4p_5+2p_6+p_7)}{48}}}{e^{\Lambda\frac{5(8p_1+12p_2+14p_3+7p_4+4p_5+2p_6+p_7)}{48}} + e^{\Lambda}}$$

Consider now the case of a trader observing a price $P = 10,000$. His expected payoffs for buying writes:

$$u_{5,B} = \frac{1}{2} \times 10 \times \mathbb{P}(B|P = 100,000) = u_{6,B}$$

The probability to buy is therefore:

$$p_5 = p_6$$

Consider now the case of a trader observing a price $P = 1,000$. His expected payoffs for buying writes:

$$\begin{aligned} u_{4,B} &= \frac{3}{7} \times 10 \times \mathbb{P}(B|P = 10,000) \\ &= 5(\frac{8p_1 + 12p_2 + 14p_3 + 7p_4 + 4p_5 + 2p_6 + p_7}{56}) \end{aligned}$$

The probability to buy is therefore:

$$p_4 = \frac{e^{\Lambda \frac{5(8p_1+12p_2+14p_3+7p_4+4p_5+2p_6+p_7)}{56}}}{e^{\Lambda \frac{5(8p_1+12p_2+14p_3+7p_4+4p_5+2p_6+p_7)}{56}} + e^{\Lambda}}$$

Consider now the case of a trader observing a price $P = 100$. His expected payoffs for buying writes:

$$u_{3,B} = \frac{3}{7} \times 10 \times \mathbb{P}(B|P = 1,000) = u_{4,B}$$

The probability to buy is therefore:

$$p_3 = p_4$$

Consider now the case of a trader observing a price $P = 10$. His expected payoffs for buying writes:

$$u_{2,B} = 1 \times 10 \times \mathbb{P}(B|P = 1,000) = 10 \frac{8p_1 + 12p_2 + 14p_3 + 7p_4 + 4p_5 + 2p_6 + p_7}{48}$$

The probability to buy is therefore:

$$p_2 = \frac{e^{\Lambda \frac{5(8p_1+12p_2+14p_3+7p_4+4p_5+2p_6+p_7)}{24}}}{e^{\Lambda \frac{5(8p_1+12p_2+14p_3+7p_4+4p_5+2p_6+p_7)}{24}} + e^{\Lambda}}$$

Consider finally the case of a trader observing a price $P = 1$. His expected payoffs for buying writes:

$$u_{1,B} = 10 \times \mathbb{P}(B|P = 10) = u_{2,B}$$

The probability to buy is therefore:

$$p_1 = p_2$$

Consequently, in equilibrium:

$$
\begin{aligned}
p_7 &= \frac{1}{1+e^{\Lambda}} \\
p_6 &= \frac{e^{\Lambda \frac{5(20p_2+21p_4+6p_6+\frac{1}{1+e^{\Lambda}})}{48}}}{e^{\Lambda \frac{5(20p_2+21p_4+6p_6+\frac{1}{1+e^{\Lambda}})}{48}}+e^{\Lambda}} \\
p_5 &= p_6 \\
p_4 &= \frac{e^{\lambda \frac{5(20p_2+21p_4+6p_6+\frac{1}{1+e^{\Lambda}})}{56}}}{e^{\lambda \frac{5(20p_2+21p_4+6p_6+\frac{1}{1+e^{\Lambda}})}{56}}+e^{\lambda}} \\
p_3 &= p_4 \\
p_2 &= \frac{e^{\Lambda \frac{5(20p_2+21p_4+6p_6+\frac{1}{1+e^{\Lambda}})}{24}}}{e^{\Lambda \frac{5(20p_2+21p_4+6p_6+\frac{1}{1+e^{\Lambda}})}{24}}+e^{\Lambda}} \\
p_1 &= p_2
\end{aligned}
$$

5.3.2 K=10,000, 2 classes (1 10) (100 1,000 10,000 100,000 1,000,000)

Consider the case where there exist two analogy classes.
Consider first the case of a trader observing a price $P = 1,000,000$. His expected payoffs for buying writes:

$$u_{7,B} = 0$$

The probability to buy is therefore:

$$p_7 = \frac{1}{1+e^{\Lambda}}$$

Consider now the case of a trader observing a price $P = 100,000$. His expected payoffs for buying writes:

$$u_{6,B} = \frac{1}{2} \times 10 \times \mathbb{P}(B|P = 1,000,000) = 5 \times (\frac{14p_3 + 7p_4 + 4p_5 + 2p_6 + p_7}{28})$$

The probability to buy is therefore:

$$p_6 = \frac{e^{\Lambda \frac{5(14p_3+7p_4+4p_5+2p_6+p_7)}{28}}}{e^{\Lambda \frac{5(14p_3+7p_4+4p_5+2p_6+p_7)}{28}}+e^{\Lambda}}$$

Consider now the case of a trader observing a price $P = 10,000$. His expected payoffs for buying writes:

$$u_{5,B} = \frac{1}{2} \times 10 \times \mathbb{P}(B|P = 100,000) = u_{6,B}$$

The probability to buy is therefore:

$$p_5 = p_6$$

Consider now the case of a trader observing a price $P = 1,000$. His expected payoffs for buying and not buying respectively write:

$$u_{4,B} = \frac{3}{7} \times 10 \times \mathbb{P}(B|P = 10,000) = \frac{30}{7} \times \frac{14p_3 + 7p_4 + 4p_5 + 2p_6 + p_7}{28}$$

The probability to buy is therefore:

$$p_4 = \frac{e^{\Lambda \frac{15(14p_3+7p_4+4p_5+2p_6+p_7)}{98}}}{e^{\Lambda \frac{15(14p_3+7p_4+4p_5+2p_6+p_7)}{98}} + e^{\Lambda}}$$

Consider now the case of a trader observing a price $P = 100$. His expected payoffs for buying and not buying respectively write:

$$u_{3,B} = \frac{3}{7} \times 10 \times \mathbb{P}(B|P = 1,000) = u_{4,B}$$

The probability to buy is therefore:

$$p_3 = p_4$$

Consider now the case of a trader observing a price $P = 10$. His expected payoffs for buying and not buying respectively write:

$$u_{2,B} = 1 \times 10 \times \mathbb{P}(B|P = 100) = 10 \times \frac{14p_3 + 7p_4 + 4p_5 + 2p_6 + p_7}{28}$$

The probability to buy is therefore:

$$p_2 = \frac{e^{\Lambda \frac{5(14p_3+7p_4+4p_5+2p_6+p_7)}{14}}}{e^{\Lambda \frac{5(14p_3+7p_4+4p_5+2p_6+p_7)}{14}} + e^{\Lambda}}$$

Consider finally the case of a trader observing a price $P = 1$. His expected payoffs for buying and not buying respectively write:

$$u_{1,B} = 10 \times \mathbb{P}(B|P = 10) = 1 \times 10 \times \frac{2p_1 + 3p_2}{5}$$

The probability to buy is therefore:

$$p_1 = \frac{e^{2\Lambda(2p_1+3p_2)}}{e^{2\Lambda(2p_1+3p_2)} + e^{\Lambda}}$$

Consequently, in equilibrium:

$$\begin{aligned}
p_7 &= \frac{1}{1+e^{\Lambda}} \\
p_6 &= \frac{e^{\Lambda\frac{5(21p_4+6p_6+p_7)}{28}}}{e^{\Lambda\frac{5(21p_4+6p_6+p_7)}{28}} + e^{\Lambda}} \\
p_5 &= p_6 \\
p_4 &= \frac{e^{\Lambda\frac{15(21p_4+6p_6+p_7)}{98}}}{e^{\Lambda\frac{15(21p_4+6p_6+p_7)}{98}} + e^{\Lambda}} \\
p_3 &= p_4 \\
p_2 &= p_2 = \frac{e^{\Lambda\frac{5(14p_3+7p_4+4p_5+2p_6+p_7)}{14}}}{e^{\Lambda\frac{5(14p_3+7p_4+4p_5+2p_6+p_7)}{14}} + e^{\Lambda}} \\
p_1 &= \frac{e^{2\Lambda(2p_1+3p_2)}}{e^{2\Lambda(2p_1+3p_2)} + e^{\Lambda}}
\end{aligned}$$

5.4 No cap

Consider now an environment in which there is no cap on the initial price. There is an infinity of possible prices. Given the probability distribution of the first price, we have:

$$\begin{aligned}
\mathbb{P}(\text{I observe } P = 1) &= \frac{1}{6} \\
\mathbb{P}(\text{I observe } P = 10) &= \frac{1}{4} \\
\mathbb{P}(\text{I observe } P = 10^n, n \geq 2) &= \frac{1}{3}\left((\frac{1}{2})^{n-1} + (\frac{1}{2})^{n} + (\frac{1}{2})^{n+1}\right) \\
&= (\frac{1}{2})^{n-1}\frac{7}{12}
\end{aligned}$$

Besides, recall that:

$$\begin{aligned} q(.,1) &= q(.,10) = 1 \\ q(.,P &\geq 100) = q(10000,1000) = \frac{3}{7} \end{aligned}$$

Let p_1, p_2, and p_3 denote the actual probability that a trader buys after observing prices equal to 1, 10, $P \geq 100$ respectively. Let $\mathbb{P}(B|P = 1)$, $\mathbb{P}(B|P = 10)$, and $\mathbb{P}(B|P \geq 100)$ be the corresponding probabilities as (mis)perceived by traders using analogy classes. In the one-class ABEE, players bundle probabilities to buy so that:

$$\mathbb{P}(B|P) = \frac{\frac{1}{6}p_1 + \frac{1}{4}p_2 + \left(1 - \frac{1}{6} - \frac{1}{4}\right)p_3}{\frac{1}{6} + \frac{1}{4} + \left(1 - \frac{1}{6} - \frac{1}{4}\right)} = \frac{2p_1 + 3p_2 + 7p_3}{12}, \forall P$$

while in the two-class ABEE,

$$\begin{aligned} \text{Class II} &: \quad \mathbb{P}(B|P \geq 100) = \frac{\left(1 - \frac{1}{6} - \frac{1}{4}\right)p_3}{1 - \frac{1}{6} - \frac{1}{4}} = p_3 \\ \text{Class I} &: \quad \mathbb{P}(B|P = 1, 10) = \frac{\frac{1}{6}p_1 + \frac{1}{4}p_2}{\frac{1}{6} + \frac{1}{4}} = \frac{2p_1 + 3p_2}{5}. \end{aligned}$$

5.4.1 No cap, 1 class

Consider the case where there exists only one analogy class.
Consider first the case of a trader observing a price $P \geq 100$. His expected payoffs for buying and not buying respectively write:

$$u_{3+,B} = \frac{3}{7} \times 10 \times \mathbb{P}(B|P \geq 100) = \frac{30}{7} \times \frac{2p_1 + 3p_2 + 7p_3}{12}$$

The probability to buy is therefore

$$p_3 = \frac{e^{\Lambda \times \frac{5}{14}(2p_1+3p_2+7p_3)}}{e^{\Lambda \times \frac{5}{14} \times (2p_1+3p_2+7p_3)} + e^{\Lambda}}$$

Consider now the case of a trader observing a price $P = 10$. His expected payoffs for buying and not buying respectively write:

$$u_{2,B} = 10 \times \mathbb{P}(B|P \geq 100) = 10 \times \frac{2p_1 + 3p_2 + 7p_3}{12}$$

The probability to buy is therefore

$$p_2 = \frac{e^{\Lambda \frac{5}{6}(2p_1+3p_2+7p_3)}}{e^{\Lambda \frac{5}{6}(2p_1+3p_2+7p_3)} + e^{\Lambda}}$$

Consider finally the case of a trader observing a price $P = 1$. His expected payoffs for buying and not buying respectively write:

$$u_{1,B} = 10 \times \mathbb{P}(B|P = 10) = u_{2,B}$$

The probability to buy is therefore

$$p_1 = p_2$$

Consequently, in equilibrium:

$$\begin{aligned} p_3 &= \frac{e^{\Lambda \times \frac{5}{14}(5p_2+7p_3)}}{e^{\Lambda \times \frac{5}{14} \times (5p_2+7p_3)} + e^{\Lambda}} \\ p_2 &= \frac{e^{\Lambda \frac{5}{6}(5p_2+7p_3)}}{e^{\Lambda \frac{5}{6}(5p_2+7p_3)} + e^{\Lambda}} \\ p_1 &= p_2 \end{aligned}$$

5.4.2 No cap, 2 classes (1 10)(all others)

Consider now the case where there exist two analogy classes.
Consider first the case of a trader observing a price $P \geq 100$. His expected payoffs for buying writes:

$$u_{3+,B} = \frac{3}{7} \times 10 \times \mathbb{P}(B|P \geq 100) = \frac{30}{7} \times p_3$$

The probability to buy is therefore :

$$p_3 = \frac{e^{\Lambda \times \frac{30}{7} \times p_3}}{e^{\Lambda \times \frac{30}{7} \times p_3} + e^{\Lambda}}$$

Consider now the case of a trader observing a price $P = 10$. His expected payoffs for buying writes:

$$u_{2,B} = 10 \times \mathbb{P}(B|P \geq 100) = 10 \times p_3$$

The probability to buy is therefore:

$$p_2 = \frac{e^{\Lambda 10\times p_3}}{e^{\Lambda 10\times p_3} + e^{\Lambda}}$$

Consider finally the case of a trader observing a price $P = 1$. His expected payoffs for buying writes:

$$u_{1,B} = 10 \times \mathbb{P}(B|P = 10) = 10 \times \frac{2p_1 + 3p_2}{5}$$

The probability to buy is therefore

$$p_1 = \frac{e^{\Lambda(4p_1+6p_2)}}{e^{\Lambda(4p_1+6p_2)} + e^{\Lambda}}$$

Consequently, in equilibrium:

$$\begin{aligned}
p_3 &= \frac{e^{\Lambda\times\frac{30}{7}\times p_3}}{e^{\Lambda\times\frac{30}{7}\times p_3} + e^{\Lambda}} \\
p_2 &= \frac{e^{\Lambda 10\times p_3}}{e^{\Lambda 10\times p_3} + e^{\Lambda}} \\
p_1 &= \frac{e^{\Lambda(4p_1+6p_2)}}{e^{\Lambda(4p_1+6p_2)} + e^{\Lambda}}
\end{aligned}$$

5.5 An extension of the Analogy-Based Expectations Equilibrium model with Heterogeneous Quantal Response

We now extend the ABEE model with QR to an ABEE model with Heterogeneous Quantal Response (below HABEE). We again consider the two types of analogy-based expectations equilibria, the one in which there is only one analogy class, and the one in which there are two classes, Class I including traders who are sure not to be last in the market sequence, and Class II including the remaining traders.

We derive the conditional probabilities to buy for risk-neutral traders observing prices of $P \in \{1, 10...\}$ in this HABEE model. Let $u_{k,B}$ be the expected payoff of risk-neutral player observing $P = P_k$ if he buys, $u_\varnothing$ his expected payoff if he does not buy.

In the heterogeneous quantal response model, the probability with which the trader buys conditional on observing P writes:

$$\mathbb{P}_i\left(B|P=P_k\right)=\frac{e^{\lambda_i u_{k,B}}}{e^{\lambda_i u_{k,B}}+e^{\lambda_i u_{\varnothing}}},$$

where λ_i is drawn from a commonly known distribution, $F_i(\lambda_i)$. As in the HQRE model, we assume that the distribution $F_i(\lambda_i)$ is common knowledge, but traders' type, λ_i , is private information known only to i. We assume that F_i is uniform $[\Lambda-\frac{\epsilon}{2},\Lambda+\frac{\epsilon}{2}]$. For computational reasons, we discretize this interval with a tick size t, therefore $f(\lambda_i)=\frac{1}{\frac{\epsilon}{t}+1}=f$, and there exist $\frac{\epsilon}{t}+1$ types of traders.

Let p_k^i denote the actual probability that a type i trader buys after observing prices equal to $P=10^k$. Let p_k denote the average probability that a trader buys after observing prices equal to $P=10^k$, and $\mathbb{P}(B|P=10^k)$ the corresponding probabilities as (mis)perceived by traders using analogy classes.

Trader i's utility if he buys depends on the price $P=P_k$ he observes but not on his type:

$$u_{k,B}=10\times q(K,10^k)\mathbb{P}(B|P=10^k)$$

Consequently, the probability to buy of a type i trader writes

$$p_k^i=\frac{e^{\lambda_i 10\times q(K,10^k)\mathbb{P}(B|P=10^k)}}{e^{\lambda_i 10\times q(K,10^k)\mathbb{P}(B|P=10^k)}+e^{\lambda_i}}$$

which yields:

$$\begin{aligned} p_k &= \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} p_k^i\mathbb{P}(\lambda_i=\lambda) \\ &= f\sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}}\frac{e^{\lambda 10\times q(K,10^k)\mathbb{P}(B|P=10^k)}}{e^{\lambda 10\times q(K,10^k)\mathbb{P}(B|P=10^k)}+e^{\lambda_i}}\end{aligned}$$

Probabilities to buy at each level of price $P=10^k$ are therefore the solution to a more complex system of equations than in the ABEE model without heterogeneity since we now have to compute a sum of exponential functions when traders' types are heterogeneous. However the link between the ABEE with QR and the ABEE with HQR is straightforward. To illustrate this, we present below as an example the case where $K = 1$.

Consider an environment in which there is a cap $K = 1$ on the initial price. There are three possible prices. Recall that given the probability distribution of the first price, we have:

$$\begin{aligned} P(\text{I observe } 1) &= \frac{1}{3} \\ P(\text{I observe } 10) &= \frac{1}{3} \\ P(\text{I observe } 100) &= \frac{1}{3} \end{aligned}$$

In the one-class ABEE, players bundle probabilities to buy so that:

$$\mathbb{P}(B|P=1) = \mathbb{P}(B|P=10) = \mathbb{P}(B|P=100) = \frac{\frac{1}{3}p_1 + \frac{1}{3}p_2 + \frac{1}{3}p_3}{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = \frac{p_1+p_2+p_3}{3},$$

while in the two-class ABEE,

$$\begin{aligned} \text{Class II} &: \quad \mathbb{P}(B|P=100) = p_3 \\ \text{Class I} &: \quad \mathbb{P}(B|P=1) = \mathbb{P}(B|P=10) = \frac{\frac{1}{3}p_1 + \frac{1}{3}p_2}{\frac{1}{3}+\frac{1}{3}} = \frac{p_1+p_2}{2}. \end{aligned}$$

5.5.1 K=1, 1 class HABEE

Consider the case where there exists only one analogy class.
Consider first the case of trader i observing a price $P = 100$. This trader perfectly infers from this observation that he is third in the sequence, that is, $q(1, 100) = 0$. Consequently, his expected payoffs if he buys writes:

$$u_{3,B} = 0,$$

so that his probability to buy is:

$$p_3^i = \frac{1}{1+e^{\lambda_i}}$$

Given the distribution of type-i players, the average probability to buy is:

$$\begin{aligned} p_3 &= \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} p_3^i \mathbb{P}(\lambda_i = \lambda) \\ &= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda}} \end{aligned}$$

Comparing with the model ABEE with QR, recall that we previously had obtained:

$$p_3 = \frac{1}{1+e^{\lambda}}$$

Consider now the case of trader i observing a price $P = 10$. Given that he knows that he is second, that is, $q(1,10) = 1$, his expected payoffs if he buys writes:

$$u_{2,B} = 10 \times \mathbb{P}(B|P = 100) = 10(\frac{p_1 + p_2 + p_3}{3})$$

His probability to buy is therefore:

$$p_2^i = \frac{e^{\lambda_i \frac{10(p_1+p_2+p_3)}{3}}}{e^{\lambda_i \frac{10(p_1+p_2+p_3)}{3}} + e^{\lambda_i}}$$

Given the distribution of type-i players, the average probability to buy is:

$$\begin{aligned} p_2 &= \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} p_2^i \mathbb{P}(\lambda_i = \lambda) \\ &= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{\lambda \frac{10(p_1+p_2+p_3)}{3}}}{e^{\lambda \frac{10(p_1+p_2+p_3)}{3}} + e^{\lambda}} \end{aligned}$$

Comparing with the model ABEE with QR, recall that we previously had obtained:

$$p_2 = \frac{e^{\lambda \frac{10(p_1+p_2+p_3)}{3}}}{e^{\lambda \frac{10(p_1+p_2+p_3)}{3}} + e^{\lambda}}$$

Consider finally the case of trader i observing a price $P = 1$. Given that he knows that he is first, that is, $q(1,1) = 1$, his expected payoffs if he buys writes:

$$u_{1,B} = 10 \times \mathbb{P}(B|P = 10) = \frac{10}{3}(p_1 + p_2 + p_3)$$

The probability to buy is therefore:

$$p_1^i = p_2^i$$

Given the distribution of type-i players, the average probability to buy is:

$$p_1 = p_2$$

Consequently, in equilibrium:

$$\begin{aligned}
p_3 &= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda}} \\
p_2 &= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{\lambda \frac{10(2p_2+p_3)}{3}}}{e^{\lambda \frac{10(2p_2+p_3)}{3}} + e^{\lambda}} \\
p_1 &= p_2
\end{aligned}$$

Solving this system enables us to find, for each j, p_j as a function of Λ and ϵ.

Comparing with the model ABEE with QR, recall that we previously had obtained:

$$\begin{aligned}
p_3 &= \frac{1}{1+e^{\lambda}} \\
p_2 &= \frac{e^{\lambda \frac{10(2p_2+p_3)}{3}}}{e^{\lambda \frac{10(2p_2+p_3)}{3}} + e^{\lambda}} \\
p_1 &= p_2
\end{aligned}$$

5.5.2 K=1, HABEE with 2 classes (1 10)(100)

Consider now the case where there exists two analogy classes I and II. Consider first the case of trader i observing a price $P = 100$. Given that $q(1,100) = 0$, his expected payoffs for buying writes:

$$u_{3,B} = 0$$

His probability to buy is therefore:

$$p_3^i = \frac{1}{1+e^{\lambda_i}}$$

Given the distribution of type-i players, the average probability to buy is:

$$\begin{aligned} p_3 &= \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} p_3^i \mathbb{P}(\lambda_i = \lambda) \\ &= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda}} \end{aligned}$$

Consider now the case of trader i observing a price $P = 10$. Given that $q(1,10) = 1$, his expected payoffs for buying write:

$$u_{2,B} = 10 \times \mathbb{P}(B|P = 100) = 10p_3$$

The probability to buy is therefore:

$$p_2^i = \frac{e^{\lambda_i 10 p_3}}{e^{\lambda_i 10 p_3} + e^{\lambda_i}}$$

Given the distribution of type-i players, the average probability to buy is:

$$\begin{aligned} p_2 &= \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} p_2^i \mathbb{P}(\lambda_i = \lambda) \\ &= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{\lambda 10 p_3}}{e^{\lambda 10 p_3} + e^{\lambda}} \end{aligned}$$

Consider finally the case of trader i observing a price $P = 1$. Given that $q(1,1) = 1$, his expected payoffs for buying writes:

$$u_{1,B} = 10 \times \mathbb{P}(B|P = 10) = 10(\frac{p_1 + p_2}{2})$$

The probability to buy is therefore:

$$p_1^i = \frac{e^{\lambda_i 5(p_1+p_2)}}{e^{\lambda_i 5(p_1+p_2)} + e^{\lambda_i}}$$

Given the distribution of type-i players, the average probability to buy is:

$$\begin{aligned}
p_1 &= \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} p_1^i \mathbb{P}(\lambda_i = \lambda) \\
&= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{\lambda 5(p_1+p_2)}}{e^{\lambda 5(p_1+p_2)} + e^{\lambda}}
\end{aligned}$$

Consequently, in equilibrium:

$$\begin{aligned}
p_3 &= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{1}{1+e^{\lambda}} \\
p_2 &= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{\lambda 10 p_3}}{e^{\lambda 10 p_3} + e^{\lambda}} \\
p_1 &= f \sum_{\lambda=\Lambda-\frac{\epsilon}{2}}^{\Lambda+\frac{\epsilon}{2}} \frac{e^{5\lambda(p_1+p_2)}}{e^{5\lambda(p_1+p_2)} + e^{\lambda}}
\end{aligned}$$

Comparing with the model ABEE with QR, recall that we previously had obtained:

$$\begin{aligned}
p_3 &= \frac{1}{1+e^{\lambda}} \\
p_2 &= \frac{e^{\lambda 10 p_3}}{e^{\lambda 10 p_3} + e^{\lambda}} \\
p_1 &= \frac{e^{5\lambda(p_1+p_2)}}{e^{5\lambda(p_1+p_2)} + e^{\lambda}}
\end{aligned}$$

Consequently, it is straightforward to extend the ABEE model to account for heterogeneity.

6 Market behavior

To complement the individual behavior analysis provided in the main paper, we study the frequency as well as the magnitude of bubbles. The frequency of bubbles is defined as the proportion of replications in which the first trader accepts to buy the asset. The magnitude of bubbles is referred to as large if all three subjects accept to buy the asset, medium if the first two subjects accept, and small if only the first subject accepts. Figure 1 presents the results per treatment.

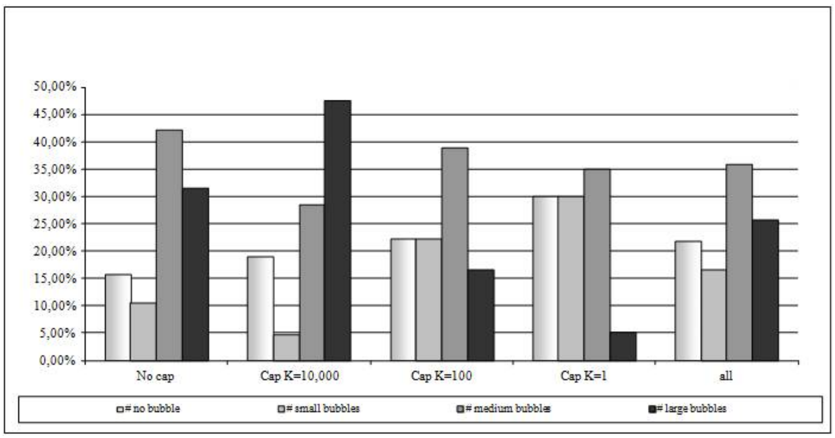

Figure 1: Probability to observe bubbles, depending on the cap on the initial price

First, Figure 1 shows that there are bubbles in an environment where backward induction is supposed to shut down speculation, namely when there exists a price cap. This is in line with the previous experimental literature cited in the introduction. We observe large bubbles even in situations where the existence of a cap enables some subjects to perfectly infer that they are last in the market sequence. A potential explanation is related to bounded rationality.[8] It is indeed possible that some subjects make mistakes and buy, in particular (but not only) when being offered a price of $100K$. Second, we also observe bubbles when there is no price cap, that is, when there

[8]An alternative explanation could be related to social preferences. However, extreme altruism would be required in order for a subject to be willing to not earn anything in order to let other subjects gain. We therefore do not focus on this interpretation.

exists a bubble equilibrium. However, bubbles are not forming 100% of the time. This indicates that traders fail to perfectly coordinate on the bubble equilibrium. This might be due to the risk of trading with an agent who is too risk averse to speculate. Another interpretation is that the (potential) existence of irrational traders (who may not buy when it would be rational) increases the risk of entering the bubble for rational traders. Finally, on Figure 1, it seems that bubbles form slightly more often when there is no price cap than when there is one, and that large bubbles are more frequent when there is a cap at $K = 10,000$ than when there is no cap. Section 5 of the main paper sheds more light on these aspects by studying individual behavior and by offering formal statistical tests. [9]

[9] In fact, the likelihood of large bubbles is not different between the treatment with a cap at $K = 10,000$ and the one with no cap (A Wilcoxon rank sum test indeed indicates a p-value of 0.307). On the contrary, subjects are more likely to speculate when bubbles are rational, specifically when they are sure not to be last.

7 Estimation of HABEE and OCH

This section shows the estimation results of the HABEE model and of the OCH model. It extends the analysis of individual behavior presented in subsection 5.2 in the main paper. The HABEE model nests the ABEE model. The OCH model is an extension of the CH model, in which the parameter θ measures overconfidence.

For each model, we estimate the parameters of interest using maximum likelihood methods for the entire data set as well as for each treatment separately. Confidence intervals are computed using a bootstrapping procedure based on 10,000 observations. We then choose the 2.5 and 97.5 percentile points values to construct 95% confidence intervals.

		Session				
		All	No cap	Cap K=10,000	Cap K=100	Cap K=1
Data						
	Sample size	234	57	63	54	60
	Av. probability buy	60%	67%	75%	54%	43%
OCH						
	Tau	0.5	0.4	1	3.9	1.7
	Theta	Ind.	Ind.	0	1	-2
	Av. probability buy	65%	66%	79%	48%	47%
	Log L	-143.61	-36.28	-32.87	-30.02	-36.47
	95% CI Tau	[0.3 - 1.0]	[0.4 - 18.4]	[0.4 - 1.8]	[0.4 - 4.7]	[0.2 - 3.9]
	% cases where Theta ind.	68%	37%	14%	29%	45%
	95% CI Theta when not ind.	[0 - 1]	[-2 - 0]	[0 - 1]	[0 - 2]	[-2 - 1]
HABEE - 1 class						
	Lambda	0.25	0.3	0.35	0.25	0.15
	Epsilon	0.1	0.0	0.1	0.1	0.3
	Av. probability buy	68%	72%	79%	67%	60%
	Log L	-141.32	-31.77	-31.68	-32.87	-39.99
	95% CI Lambda	[0.00 - 0.70]	[0.25 - 0.35]	[0.25 - 0.55]	[0.15 - 0.45]	[0.20 - 0.35]
	95% CI Epsilon	[0.20 - 0.35]	[0.0 - 0.2]	[0.0 - 1.1]	[0.0 - 0.3]	[0.0 - 0.7]
HABEE - 2 classes						
	Lambda	0.3	0.35	0.4	0.3	0.15
	Epsilon	0.4	0.1	0.2	0.0	0.3
	Av. probability buy	68%	72%	77%	68%	59%
	Log L	-137.46	-31.16	-30.87	-31.97	-39.44
	95% CI Lambda	[0.0 - 1.1]	[0.30 - 0.40]	[0.30 - 0.60]	[0.20 - 0.45]	[0.0 - 1.50]
	95% CI Epsilon	[0.20 - 0.55]	[0.0 - 0.2]	[0.0 - 1.0]	[0.0 - 0.2]	[0.0 - 3]

Table 1: Goodness of fit of HABEE with 1 class, HABEE with 2 classes, and OCH

The results of the OCH estimations are as follows.[10] First, subjects tend to be pretty overconfident. We illustrate the degree of overconfidence by the

[10] As shown in Section 4, in some circumstances, the parameter θ is not identifiable. As can be seen in Table I, this occurs when we estimate OCH on the no cap treatment data and on the entire data. This also happens in our bootstrapping simulations, so we also indicate the percentage of simulations in which this is the case.

difference, divided by τ, between τ and the average of the perceived average sophistication. This measures the extent to which agents underestimate the actual average level of sophistication in the population of players. Given our estimations, the degree of overconfidence is equal to 9% for the treatment with $K = 1$ (the estimated values are $\theta = -2$ and $\tau = 1.7$), 49% for the treatment with $K = 100$ (the estimated values are $\theta = 1$ and $\tau = 3.9$), 36% for the treatment with $K = 10,000$ (the estimated values are $\theta = 0$ and $\tau = 1.0$). Second, a log-likelihood ratio test shows that the OCH model does not significantly improve (p-value is 0.296 when pooling the estimates done on the sessions in which θ is identified, that is for $K = 1$, $K = 100$, and $K = 10,000$) the fit relative to the CH model (that also features overconfident agents since it corresponds to OCH with $\theta = 1$).

The results of the HABEE estimations are as follows. First, adding heterogeneity does not change our result on the comparison of the ABEE models with one and two classes. Indeed, a likelihood test for non-nested models confirms that the HABEE model with two classes performs significantly better than the HABEE model with one class (p-value of 0.008). Second, adding heterogeneity increases the log-likelihood but this increase is not statistically significant. A Vuong test for nested models indicates that the HABEE model with 1 class (resp. 2 classes) does not significantly perform better that the corresponding ABEE model with a p-value of 0.635 (resp. 0.671). Third, the parameter ϵ is very small in all sessions (between 0 and 0.4). As 0 lies in the 95% confidence interval when considering the estimations by treatment, we cannot reject the hypothesis that ϵ is equal to zero. The estimates of Λ are very similar to that of λ in the ABEE models.[11]

[11] As the individual λ_i belong to the interval $[\Lambda - \frac{\epsilon}{2}, \Lambda + \frac{\epsilon}{2}]$, it needs to be the case that $\Lambda \geq \frac{\epsilon}{2}$. Consequently, even if our estimation increments possible values of Λ by a tick of 0.1, the estimate may not be a multiple of 0.1.

8 Robustness: learning and professional experience

This section extends our analysis to study the effect of learning and professional experience. The first subsection reports an experiment where subjects play five replications of the game. The second subsection reports an experiment run with Executive MBA students at the London Business School.

8.1 Learning

In order to study how learning affects bubble formation, we run exactly the same experiment as in the main paper except for the number of replications. Subjects are now playing five replications in a stranger design (and this is common knowledge): subjects do not know with whom they are playing and it is very unlikely that they will be playing again with the same subjects. The experiment includes 66 subjects from the first year of Master in Finance at the University of Toulouse. This pool of students is very similar to the baseline experiment pool. There are four sessions with 15 or 18 subjects. Each subject participates in only one session and receives a 5-euro show-up fee. The minimum, median, maximum, and average gains in this experiment are respectively 1, 13, 41, and 16 euros (not including the show-up fee).

This experimental design is summarized in Table II.

Session	# Replications	# Subjects	cap on initial price, K	Equilibrium
13	5	15	1	no-bubble
14	5	18	100	no-bubble
15	5	15	10,000	no-bubble
16	5	18	∞	no-bubble or bubble

Table 2: Experimental design of the 5-period experiments

We start by constructing a data set that includes the baseline (one-shot) experiment and the first replication of the learning experiment. We thus have 300 observations. We run a logit regression capturing the effects of the number of steps of iterated reasoning, the degree of risk aversion and the price as in the baseline experiment, dropping the dummies capturing the cap effect as they were not significant. In contrast with the baseline regression though, we do not interact the dummies that capture the number of steps with the dummies that capture the informational content of prices (namely, whether there are sure not to be last, or not). This difference is motivated by the fact that at the last period, 100% of the 24 subjects who know they are not last and use more than three steps of iterated reasoning always buy, which would prevent us to capture the impact of the last period on the probability to buy on Variable 4 of the baseline regression. We therefore complement this regression by a second logit regression capturing the effects of the probability to be last, the degree of risk aversion and the price.

The results of both regressions are in columns I and II of Table III. The coefficient estimates and significance levels are very similar to the baseline case. Very few "step 0" enter the bubble but the proportion is not zero: 3/35. The propensity to enter bubbles increases with the number of steps of iterated reasoning. A Wald test of equality of the coefficients of the two dummies which capture the number of steps indeed indicates a p-value of 0.00. This propensity also increases when subjects know they are not last. A Wald test of equality of the coefficients of dummies which capture the fact that subjects know, or not, that they are not last reports a p-value of 0.00. In particular, when there is no price cap, the propensity to enter bubbles is very high: in this case, 100% of the 22 subjects buy the asset after receiving a price of 1 or 10.

A first look at the effect of learning in our experiment is offered by adding to the data set the fifth replication of the learning experiment. We then include in the experiment a dummy variable indicating that the observation corresponds to the fifth session and we interact this dummy with the other explanatory variables of interest, that is, either the number of steps of iterated reasoning, or whether subjects know that they are not last or not.

The results are in column III and IV of Table III. Overall, the coefficients of the fifth replication dummy variable and its interactions appear mostly negative but insignificant. This seems to indicate that the propensity to enter bubbles is not really lower during the fifth period.

To investigate further this result, we focus on the 66 subjects who participated in the five replications and we run a panel logit regression that controls for period and individual fixed-effects. We drop 30 observations corresponding to 6 subjects who always buy.[12] Our regression uses the set

[12]Out of these 6 subjects, 3 participated in the session with no cap, 1 in the session with $K = 1$ and 2 in the session with $K = 10,000$. No subject never buys.

	I. Baseline + Period 1		II. Baseline + Period 1		III. Baseline + Period 1 + Period 5		IV. Baseline + Period 1 + Period 5	
	Coefficient	p-value	Coefficient	p-value	Coefficient	p-value	Coefficient	p-value
Constant	-2.13	0.000	-1.99	0.002	-2.15	0.000	-2.11	0.000
$D_{Step=1 \text{ or } 2}$	2.67	0.000			2.59	0.000		
$D_{step>=3}$	3.64	0.000			3.55	0.000		
$D_{0<P(last)<1}$			2.62	0.000			2.57	0.000
$D_{P(last)=0}$			4.07	0.000			3.99	0.000
D_{P5}					0.35	0.300	0.40	0.248
$D_{Step=1 \text{ or } 2} \times D_{P5}$					-0.53	0.339		
$D_{step>=3} \times D_{P5}$					-0.09	0.866		
$D_{0<P(last)<1} \times D_{P5}$							-0.81	0.121
$D_{P(last)=0} \times D_{P5}$							0.30	0.670
Degree of risk aversion for consistent choices	-0.42	0.359	-0.78	0.101	-0.38	0.364	-0.56	0.207
Price	-0.00	0.102	-0.00	0.448	-0.00	0.101	-0.00	0.446
Number of observations	300		300		366		366	
Log likelihood	-162.61		-156.16		-197.40		-186.08	
Wald test								
$D_{Step=1 \text{ or } 2} = D_{step>=3}$	0.0005							
$D_{0<P(last)<1} = D_{P(last)=0}$			0.0000					

Table 3: Logit Regression on the Buy Decision

of explanatory variables detailed above, aggregating "step-0" and "step-1" observations due to the low number of "step-0" observations. In addition, we include the following variables: a dummy that indicates that a subject bought and lost at least once in a previous replication and that he or she is not sure to be last, a dummy that indicates that a subject bought and won at least once in a previous replication and that he or she is not sure to be last, and a dummy that indicates that a subject has been last and knew it at least once in a previous replication. The first two dummies are designed to capture reinforcement or belief-based learning.[13] The last dummy captures the behavior of subjects that have experienced what it means to receive the highest potential price. To capture potential wealth effects, we also include an additional control variable, namely the accumulated gains of a subject.

The results are in Table IV. As before, a subject's propensity to buy the overvalued asset significantly increases with the number of steps of iterated reasoning needed to derive the equilibrium strategy, and significantly decreases with his probability to be last and with his risk aversion. Our estimation further shows that learning has an ambiguous effect on the propensity to enter a bubble: subjects tend to speculate more after good experiences

[13]See Camerer and Ho (1999) for a theoretical and experimental analysis of learning in games.

	Coefficient	Statistic	p-value
Constant	0.59	0.38	0.707
Dummy which equals 1 when two or more steps of iterated reasoning from maximal price and not last	50.24	8.99	0.000
Dummy which equals 1 when two or more steps of iterated reasoning from maximal price and maybe last	45.29	8.11	0.000
Dummy which equals 1 when there is no cap and not last	8.52	3.15	0.002
Dummy which equals 1 when there is no cap and maybe last	2.88	1.30	0.193
Dummy which equals 1 when the subject bought and lost at least once in a previous replication and when he or she is not sure to be last	-3.15	-3.38	0.001
Dummy which takes value 1 when the subject bought and won at least once in a previous replication and when he or she is not sure to be last	2.24	1.59	0.113
Dummy which takes value 1 when the subject has been last and knew it at least once in a previous replication	2.54	1.63	0.103
Accumulated gains	-0.42	-2.86	0.004
Risk aversion	-31.07	-8.16	0.000
Dummy which takes value 1 in the 2nd period	0.11	0.12	0.903
Dummy which takes value 1 in the 3rd period	1.63	1.53	0.126
Dummy which takes value 1 in the 4th period	2.27	1.68	0.093
Dummy which takes value 1 in the 5th period	4.94	3.17	0.002
Log likelihood	-65.29		
Number of observations	300		

Table 4: Panel Logit Regression on the Buy Decision

and less after bad experiences. Overall, it is thus not clear that learning leads, at least rapidly, to the no-bubble equilibrium. Finally, it seems that those who have been confronted with the highest price may be more likely to buy when they are subsequently not sure to be last. This might be due to the fact that they realize the complexity of the game and are more ready to bet on other subjects' mistakes.

8.2 Professional experience

In order to study how experienced business people behave as far as bubble formation is concerned, we run exactly the same experiment as in the baseline case except for the origin of the subjects and for experimental incentives. Subjects are now students from the Executive MBA program at the London Business School. Instead of playing for euros, they played for fine chocolate boxes (worth 5 euros each). There is thus a five-time increase in the scale of the incentives. If a subject buys the asset, he ends up with 10 chocolate boxes if he is able to resell and 0 box if he is not. If he decides not to buy, he keeps the chocolate box. The rest of the design is exactly the same as in the baseline case (subjects played only once).[14] This experiment includes 54 subjects. There is only one session with a cap of 10.000 on the first price. The minimum, median, maximum, and average gains in this experiment are respectively 0, 1, 10, and 3.08 chocolate boxes. This experimental design is summarized in Table V.

Session	# Replications	# Subjects	cap on initial price, K	Equilibrium
17	1	54	10.000	no-bubble

Table 5: Experimental design of the experiment with LBS students

Our results are obtained thanks to a logit regression of the probability to buy the asset. We pool the 54 observations corresponding to LBS executive students with the 63 subjects from Toulouse University who played the one-shot game with a cap at 10.000. Overall we thus have 117 observations. We pool step-0 subjects with step-1 or 2 subjects as there are only 2 step-0 subjects who never buy. The explanatory variables are variables 3 and 4, a

[14] In the interest of time, we did not measure the level of risk aversion of the Executive MBA students.

dummy indicating that a subject is an executive from LBS, and two dummy interacting variables 3 and 4 with the dummy LBS. We also include as control variable the proposed price.

	Coefficient	p-value
Constant	0.79	0.417
Variable 3: $D_{step>=3}$ x $D_{0<P(last)<1}$	-0.09	0.929
Variable 4: $D_{step>=3}$ x $D_{P(last)=0}$	1.69	0.166
D_{LBS}	-0.52	0.649
Variable 3 x D_{LBS}	-0.81	0.529
Variable 4 x D_{LBS}	0.29	0.853
Price	-0.00	0.400
Log likelihood	-57.69	
Number of observations	117	

Table 6: Logit Regression on the Buy Decision, LBS students

The results are in Table VI. Overall, the behavior of LBS subjects appears very similar to the one of the other subjects as the coefficients are not statistically significant.

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